Linear Algebra Bootcamp

The 90 Minute Primer

Linear Algebra

- Study of functions/surfaces/spaces that do not bend or curve.

- Scalar multiplication and addition.

Matrices and Vectors

- Arrays or lists of numbers.

- Indexed first by row (i) then by column (j) $\mathbf{X}_{ij}$ $\mathbf{v}_i$

$$\mathbf{X} = \begin{pmatrix} 1 & 8 & 7 & -1 \\ 4 & 9 & 6 & 9 \\ -3 & -4 & 9 & 8 \\ -2 & -1 & 10 & 3 \\ 3 & -3 & 1 & 7 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 0.3 \\ -1 \\ 1.2 \\ -1 \end{pmatrix}$$

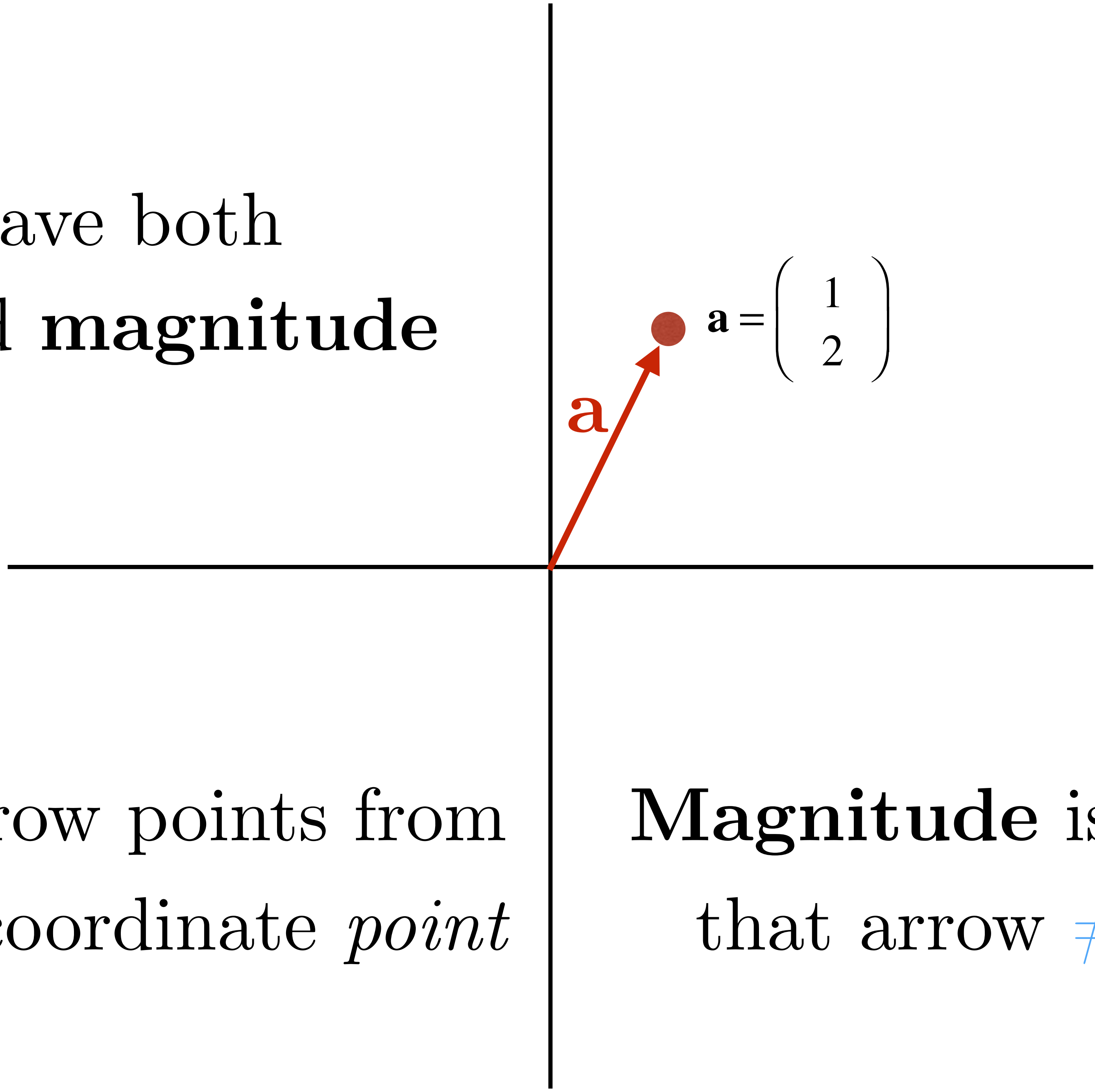
X_{23}

v_2

Vectors/Points

(Geometrically)

Vectors have both **direction** and **magnitude**



Direction arrow points from origin to the coordinate *point*

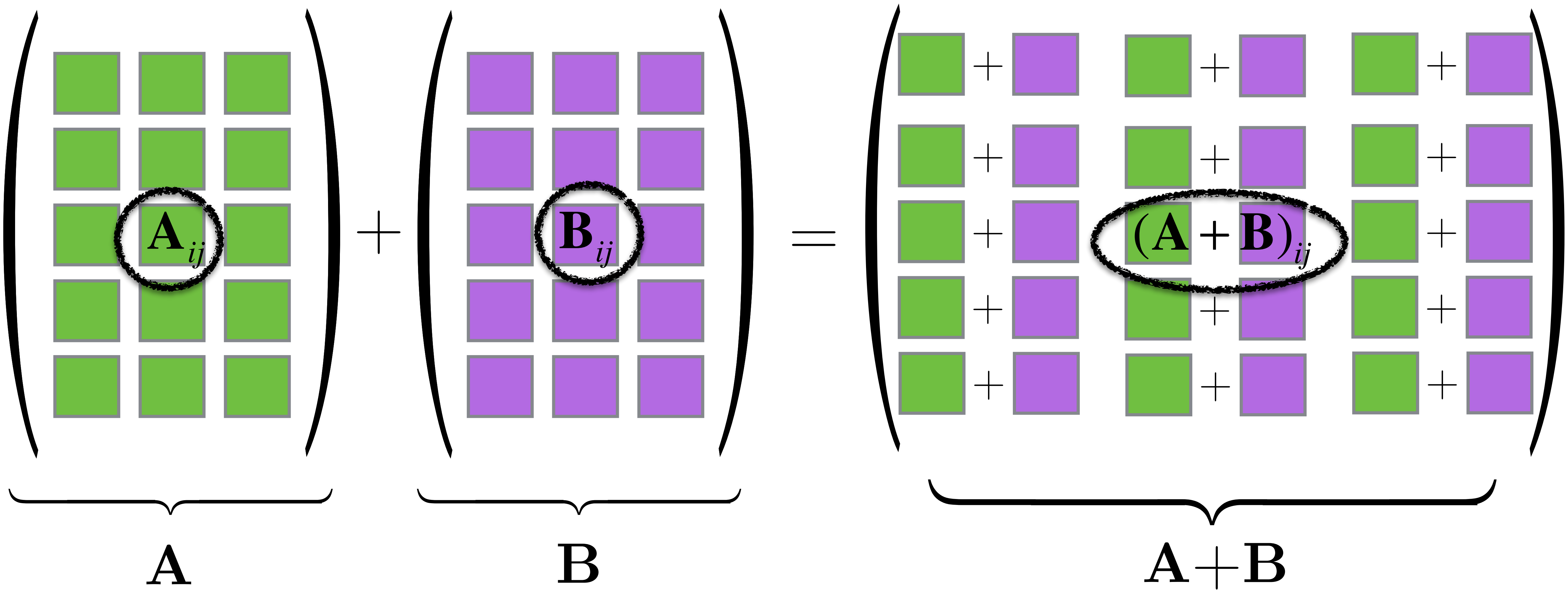
Magnitude is the length of that arrow #pythagoras

Matrix Arithmetic

(multi-dimensional math)

Addition

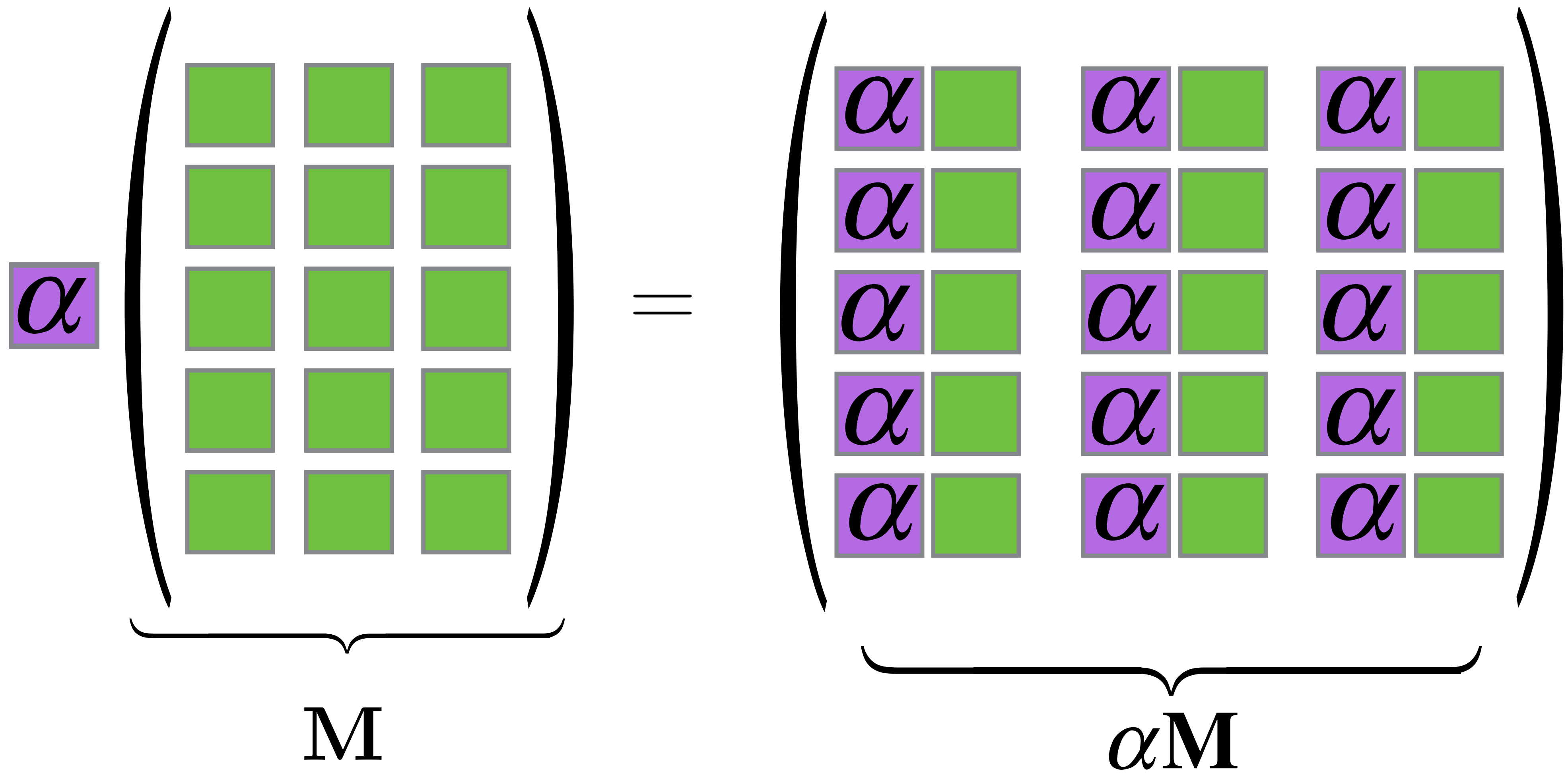
Element-wise



$$(\mathbf{A}+\mathbf{B})_{ij} = \mathbf{A}_{ij} + \mathbf{B}_{ij}$$

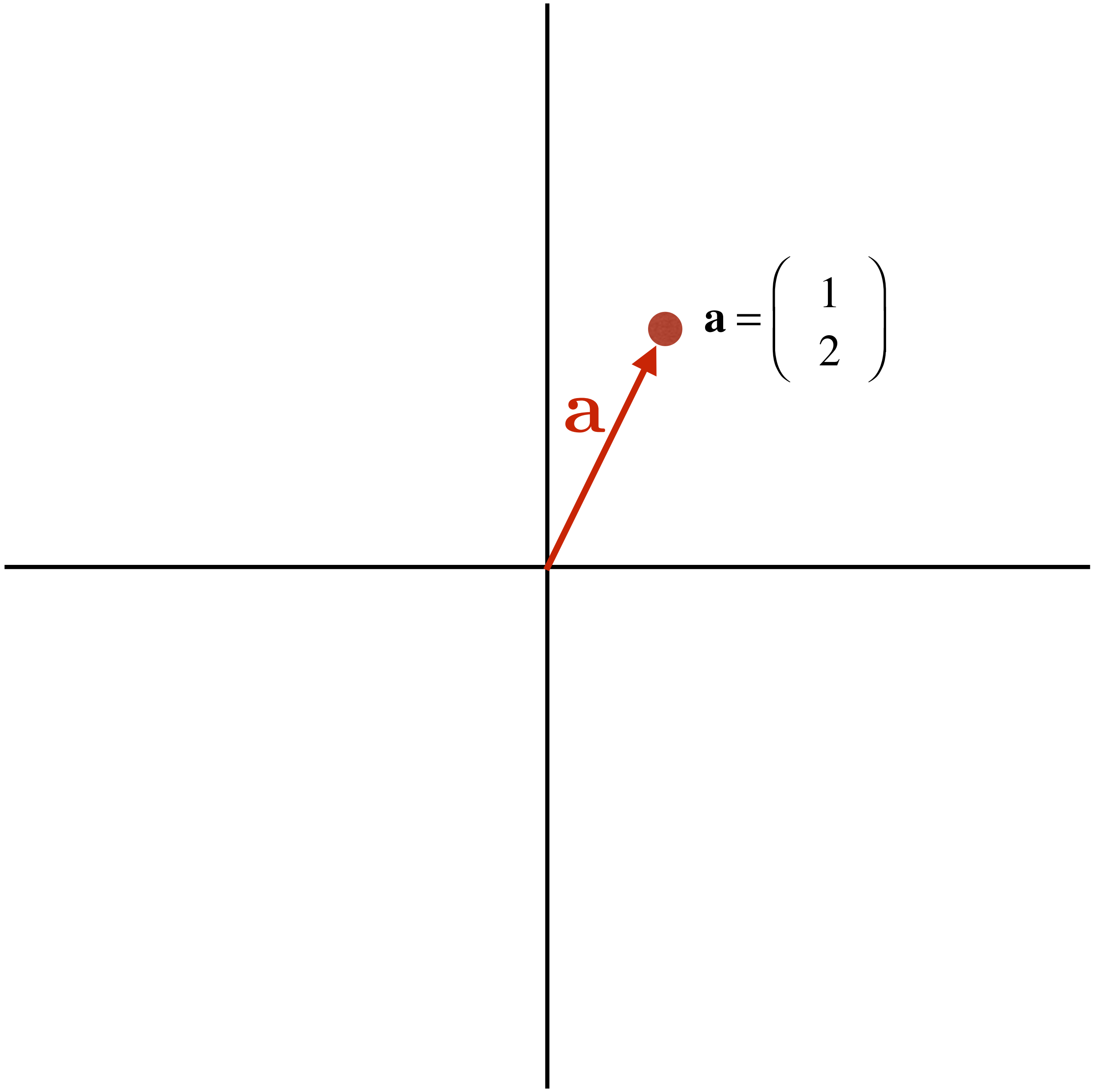
Scalar Multiplication

Element-wise



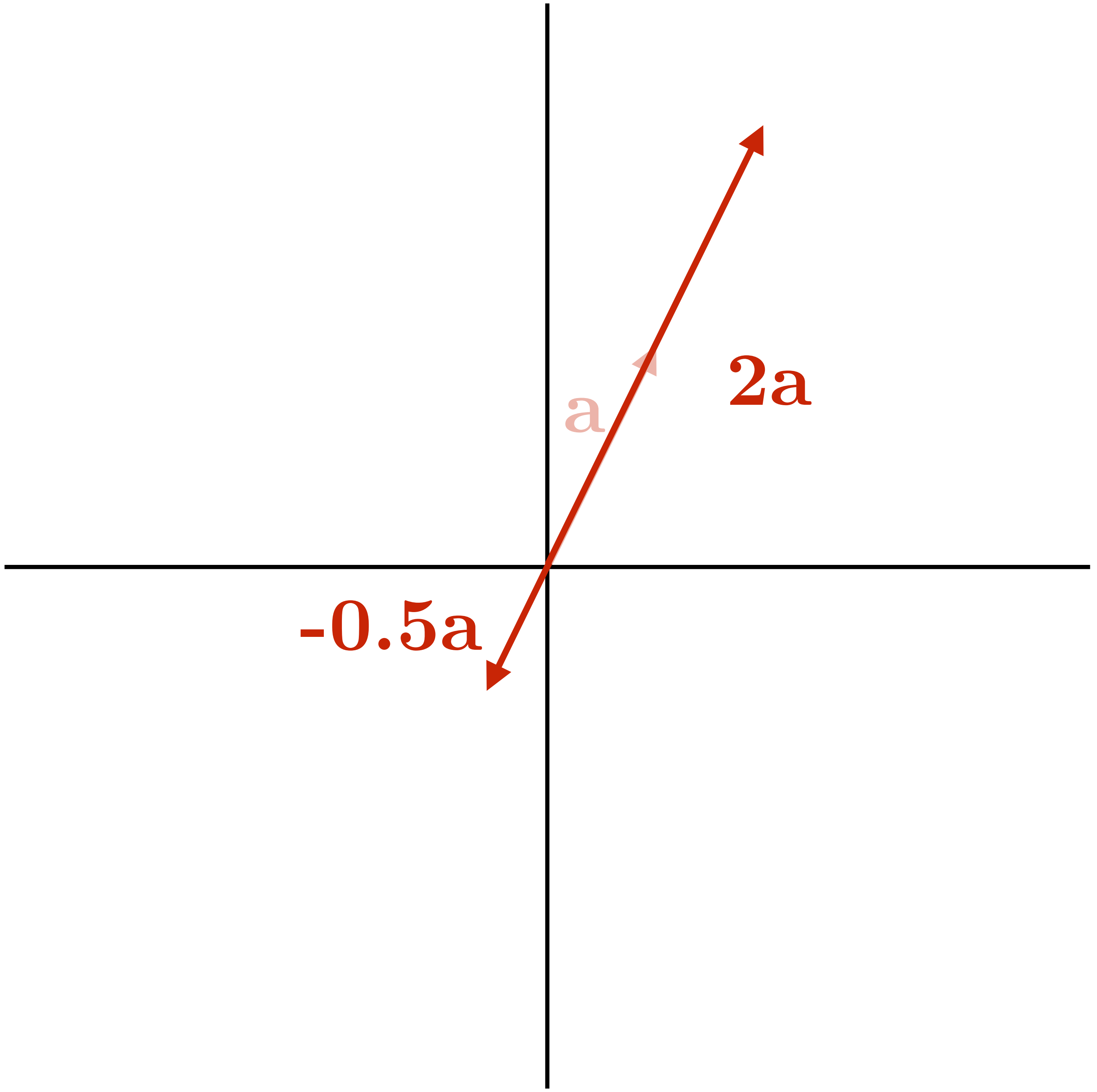
$$(\alpha \mathbf{M})_{ij} = \alpha \mathbf{M}_{ij}$$

Scalar Multiplication (Geometrically)



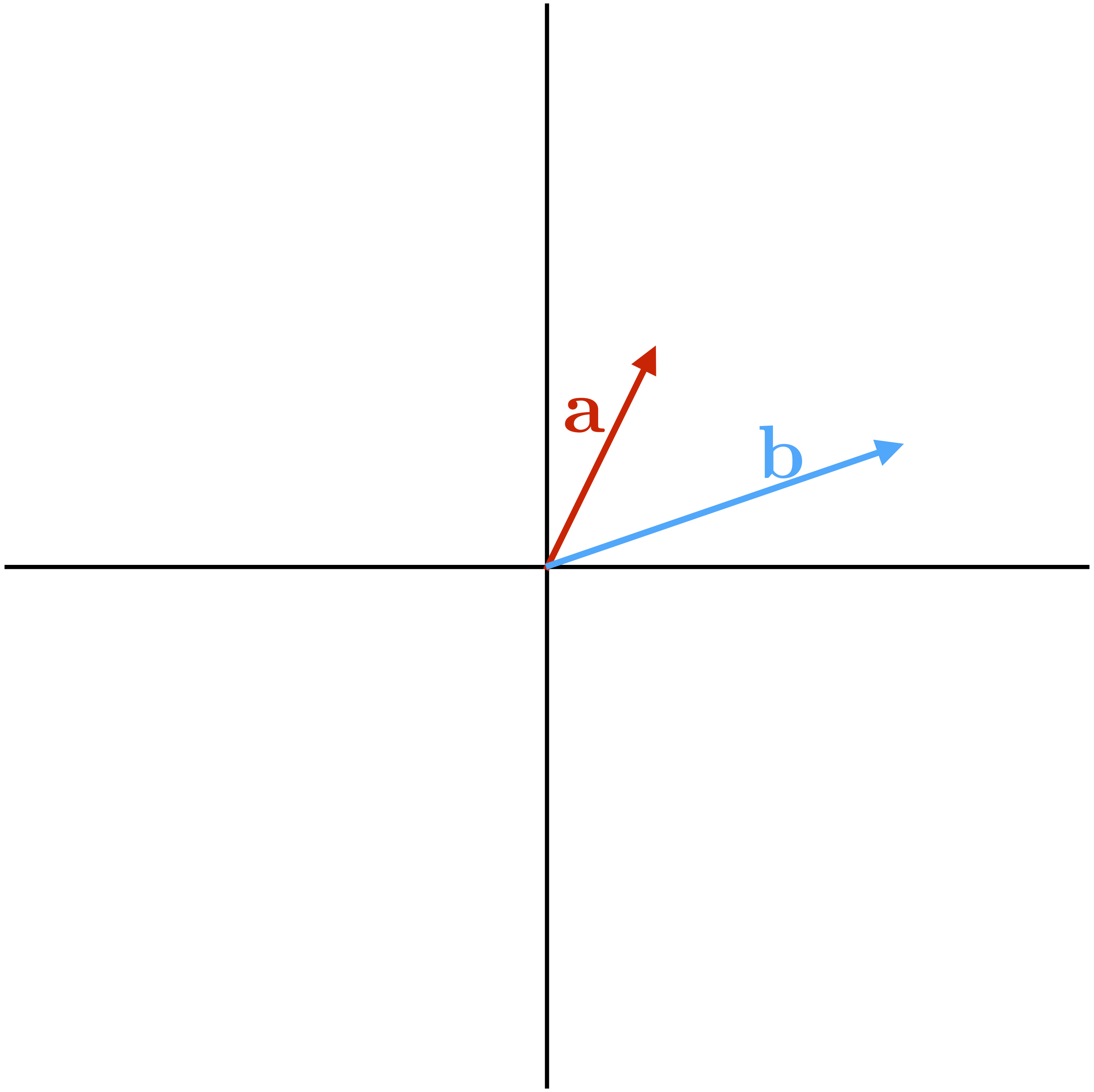
Scalar Multiplication

(Geometrically)



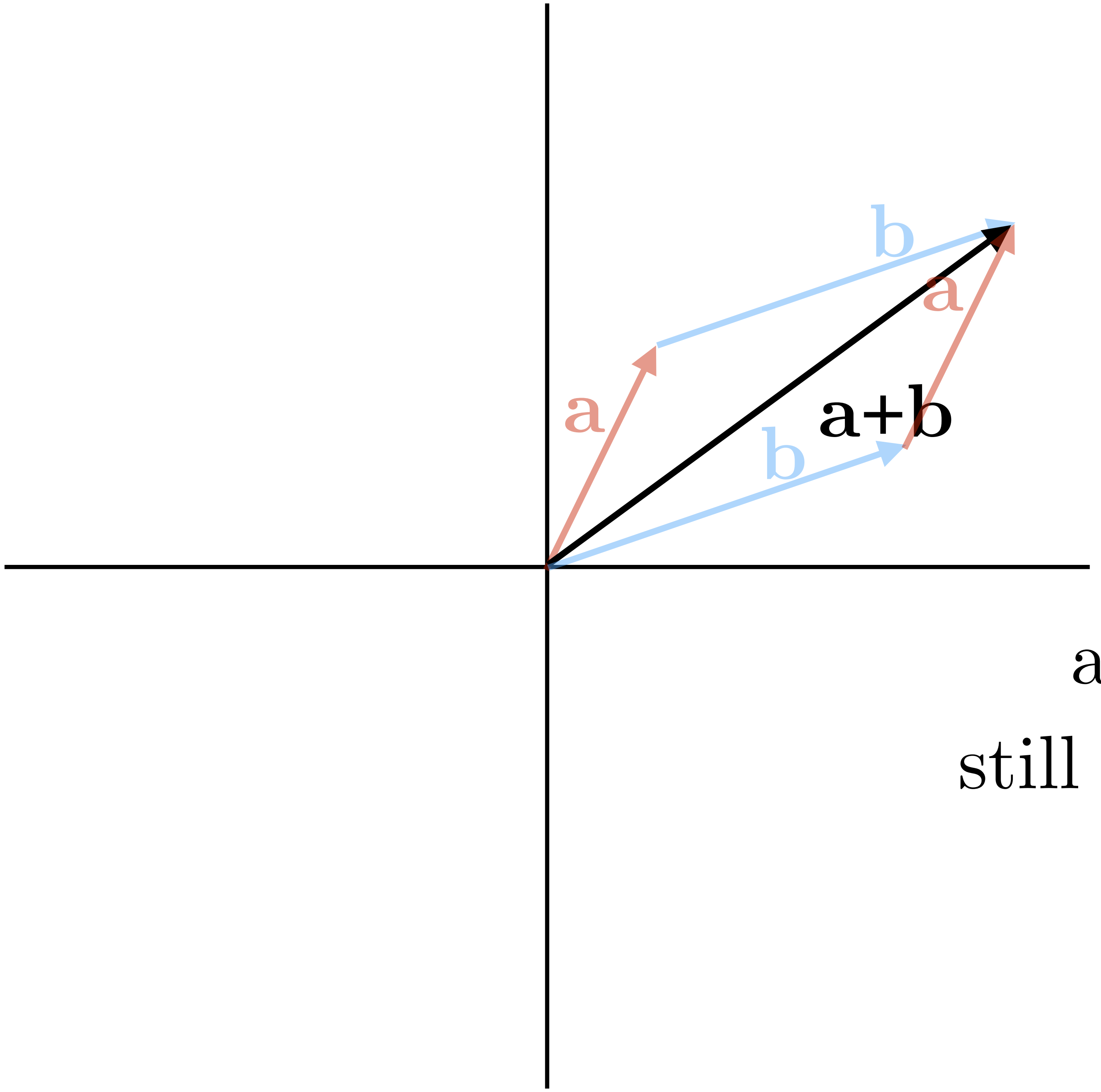
Vector Addition

(Geometrically)



Vector Addition

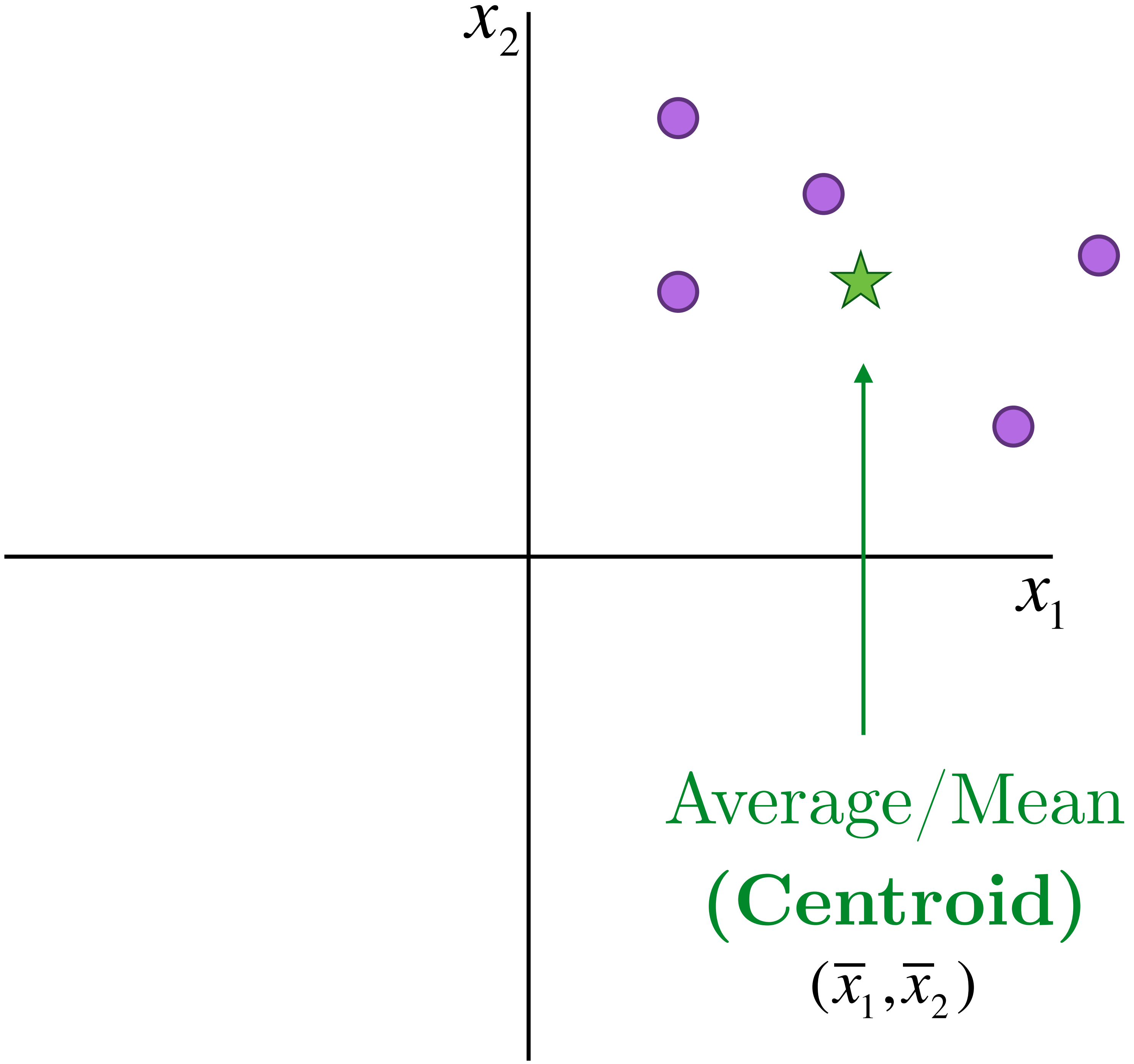
(Geometrically)



addition is
still commutative

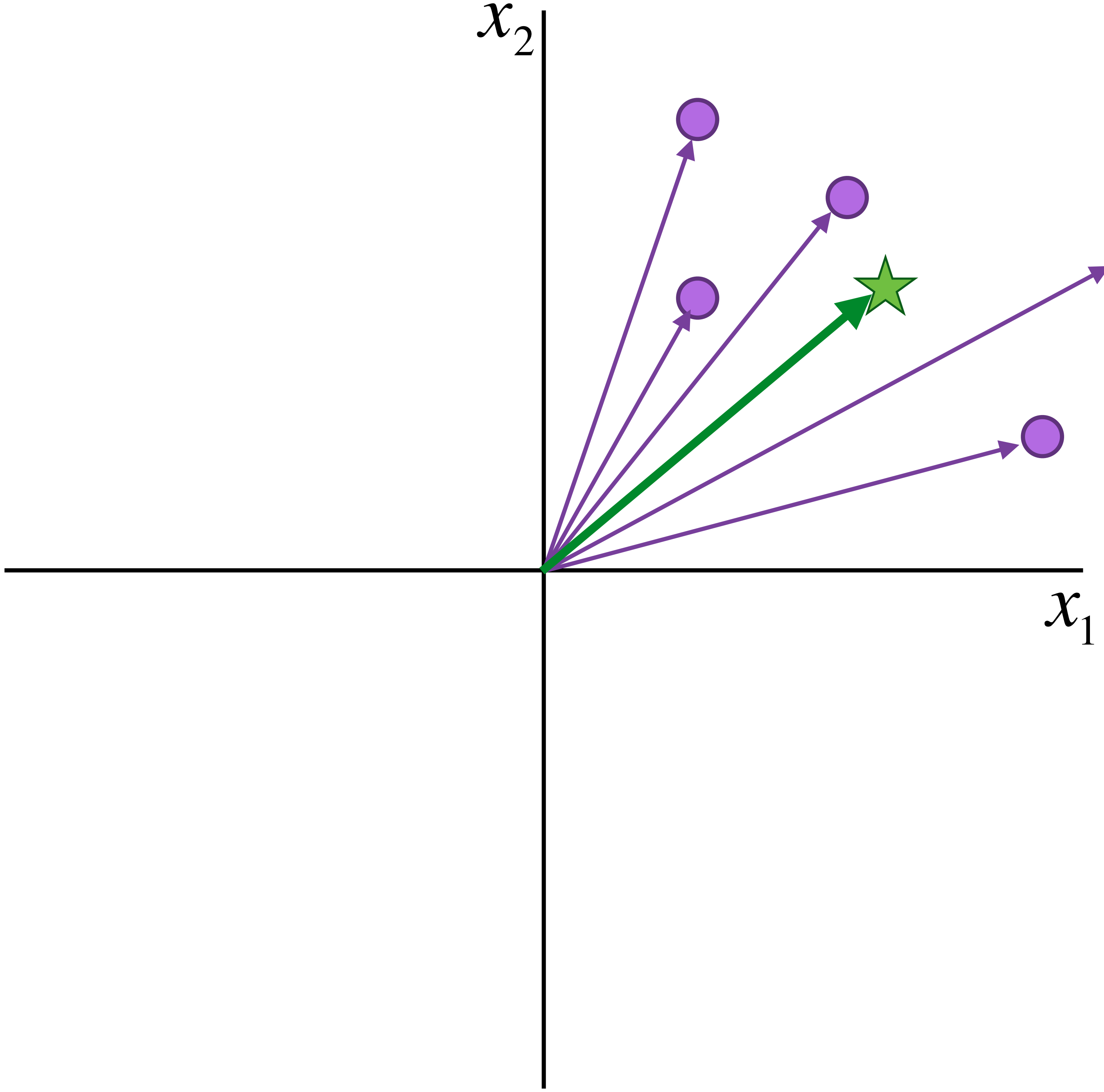
Example: Centering the data

(Geometrically)



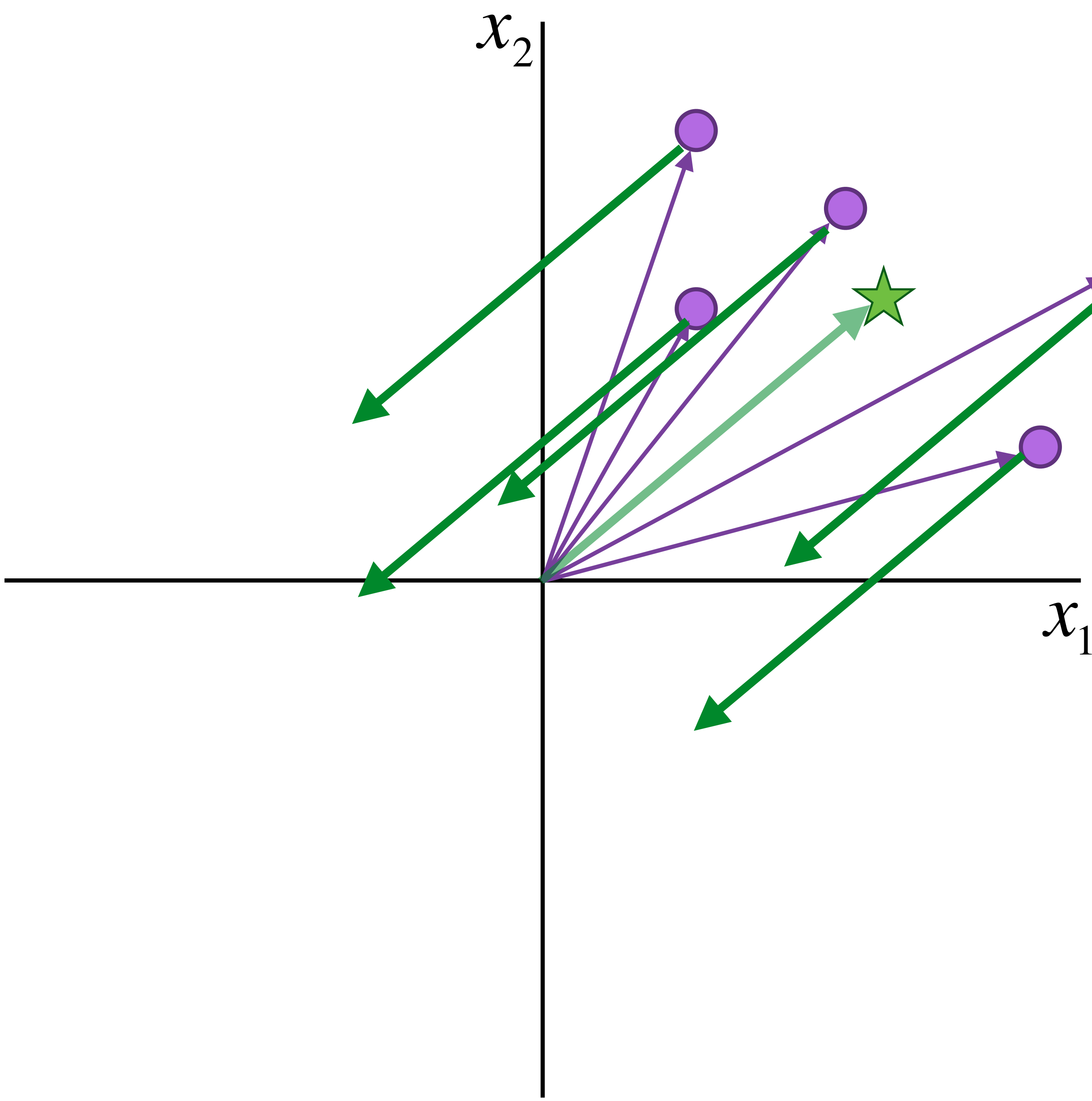
Example: Centering the data

(Geometrically)



Example: Centering the data

(Geometrically)



Example: Centering the data

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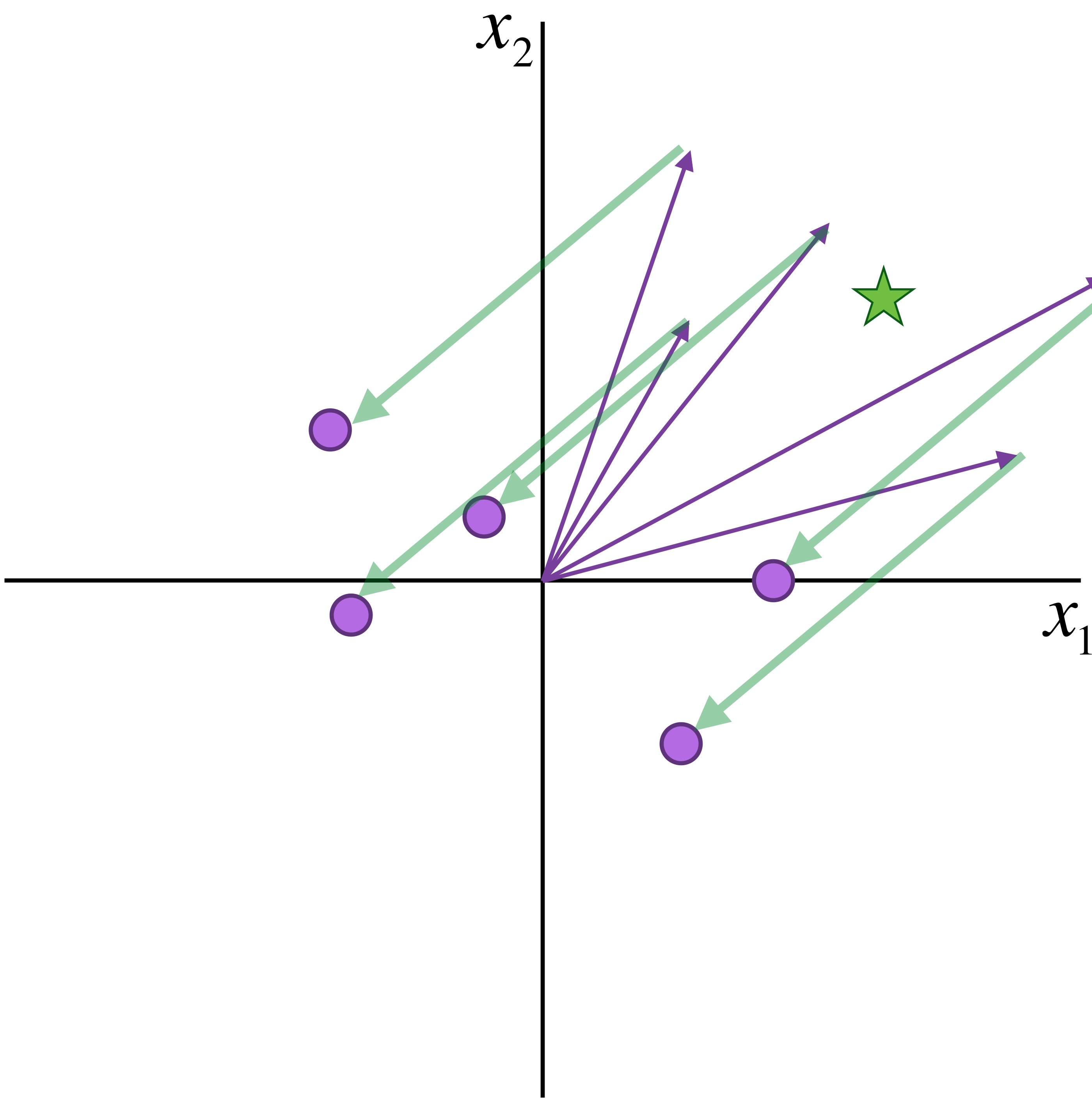
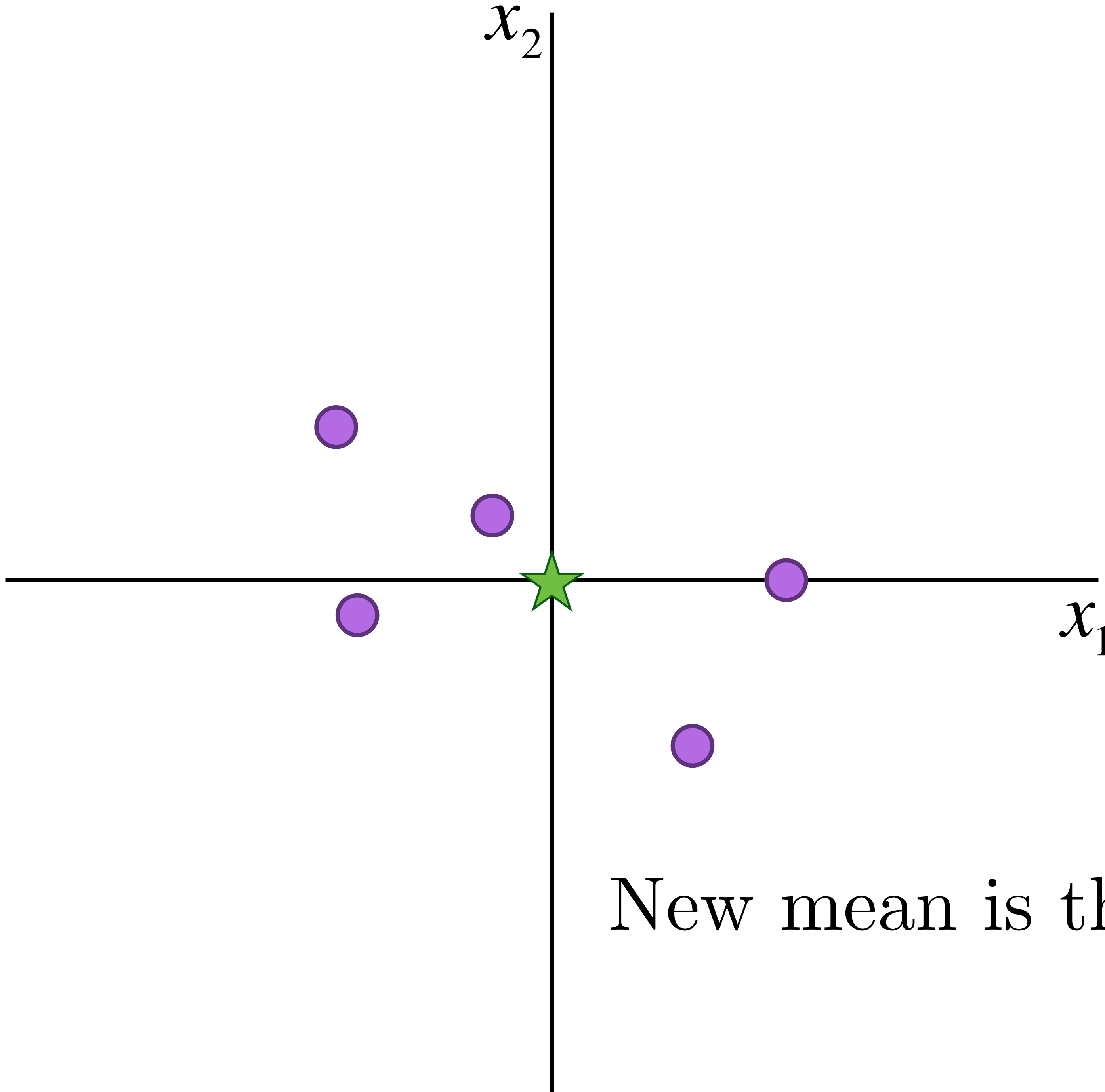


Example: Centering the data

(Geometrically)

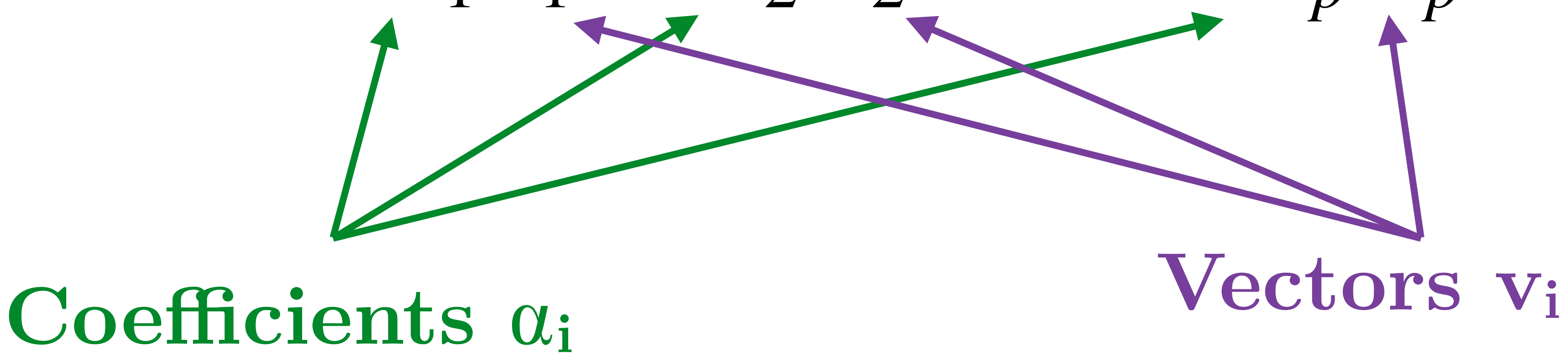


New mean is the origin (0,0)

Linear Combinations

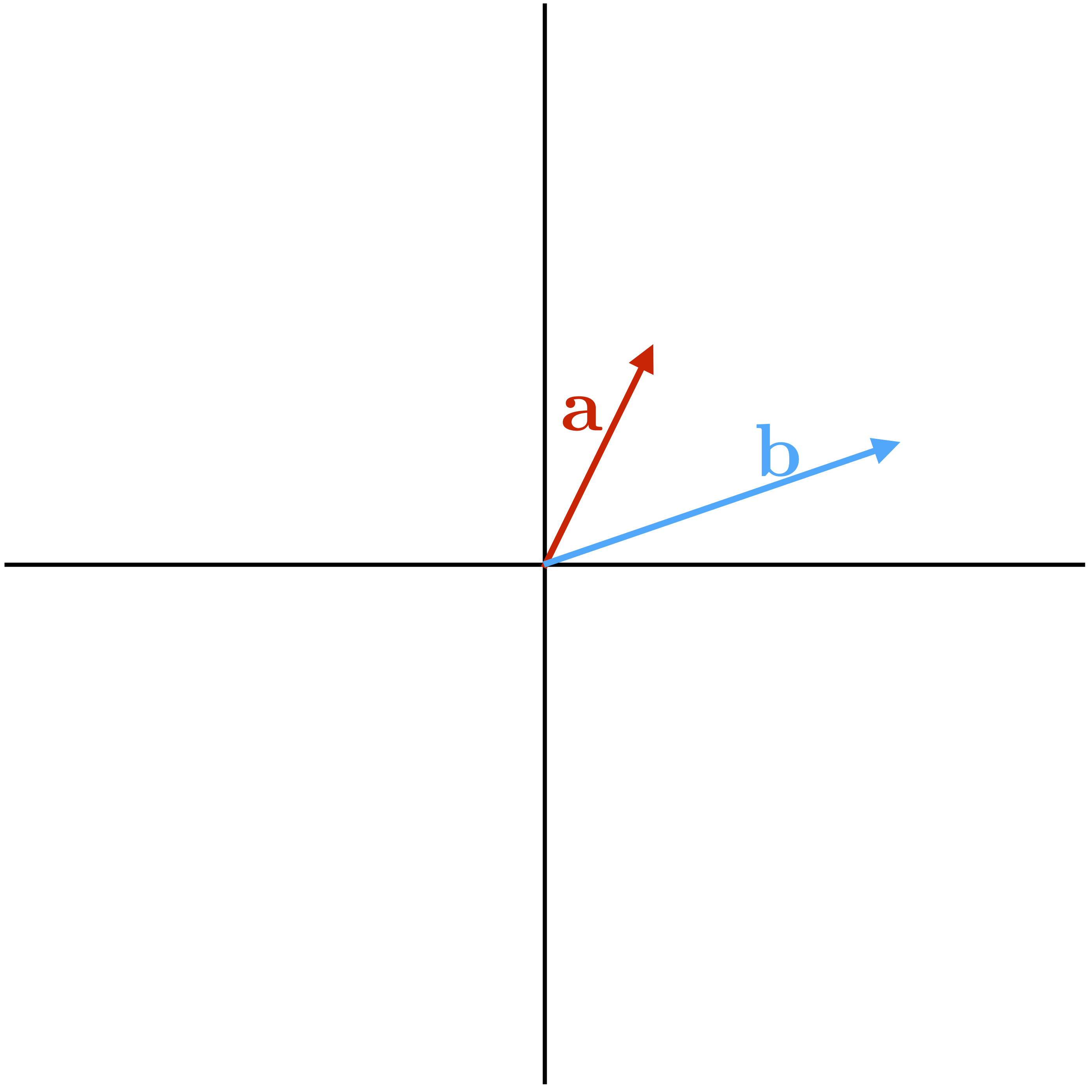
A linear combination of vectors is a just weighted sum:

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_p \mathbf{v}_p$$



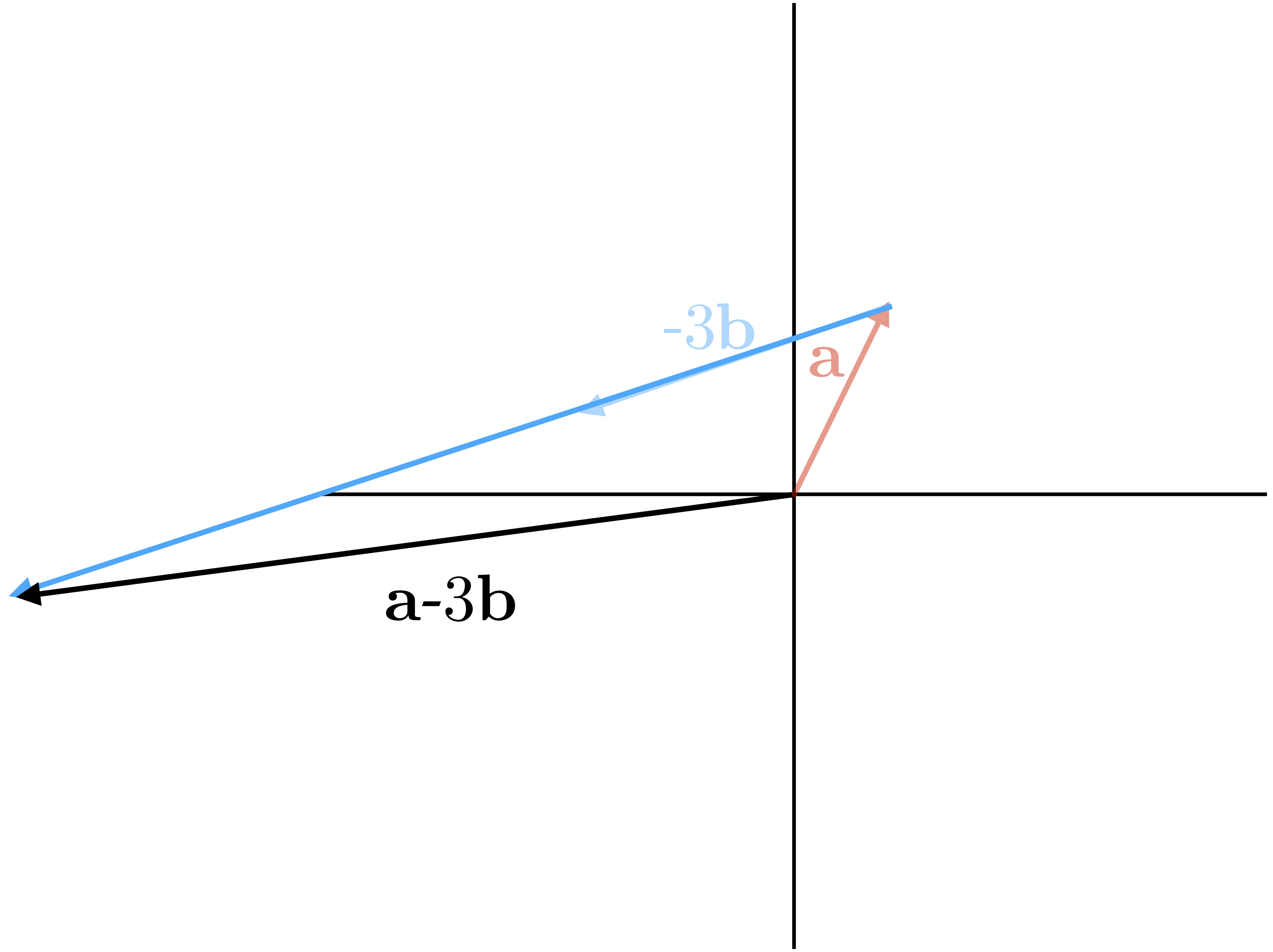
Linear Combinations

(Geometrically)



Linear Combinations

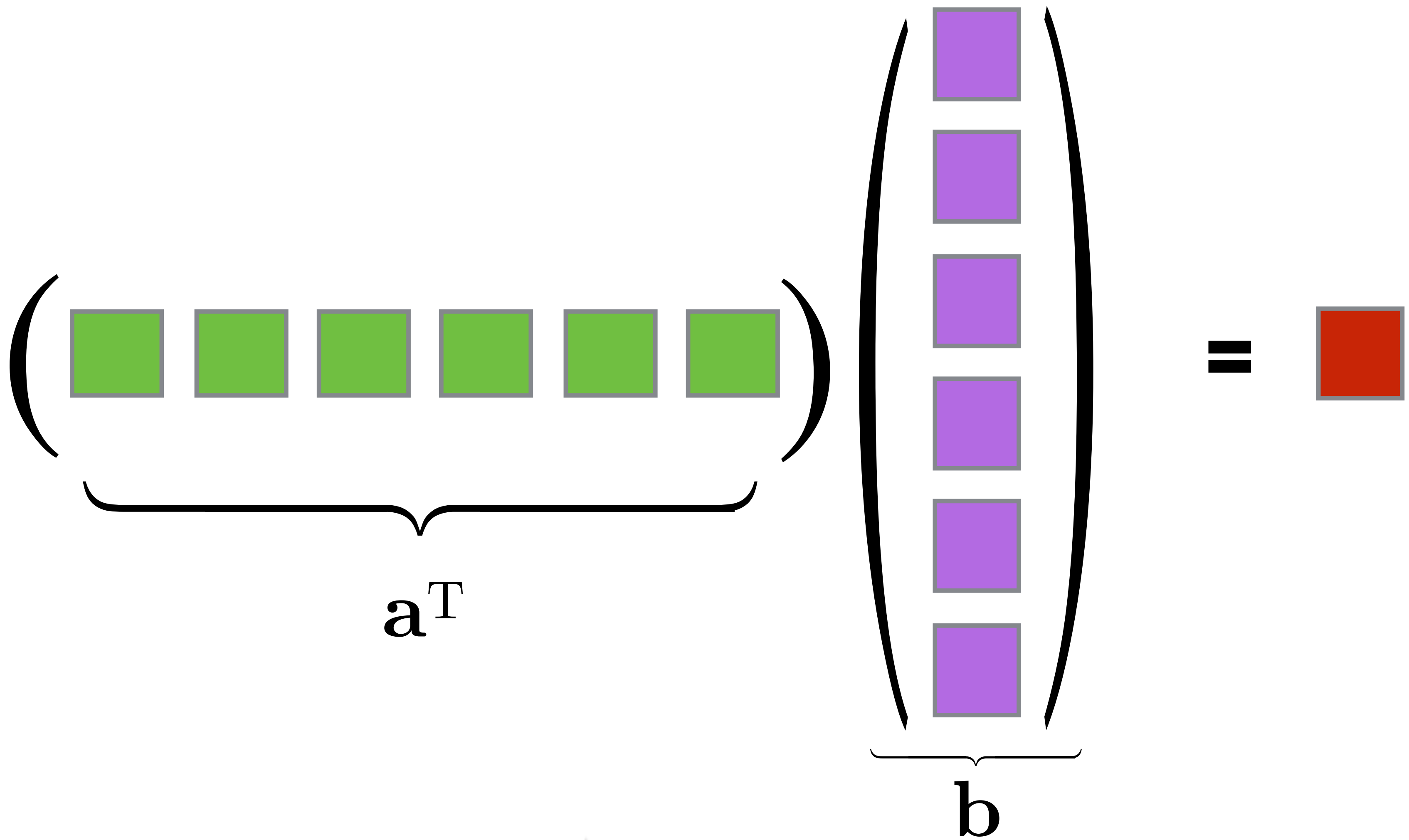
(Geometrically)



Multiplication

Inner Product

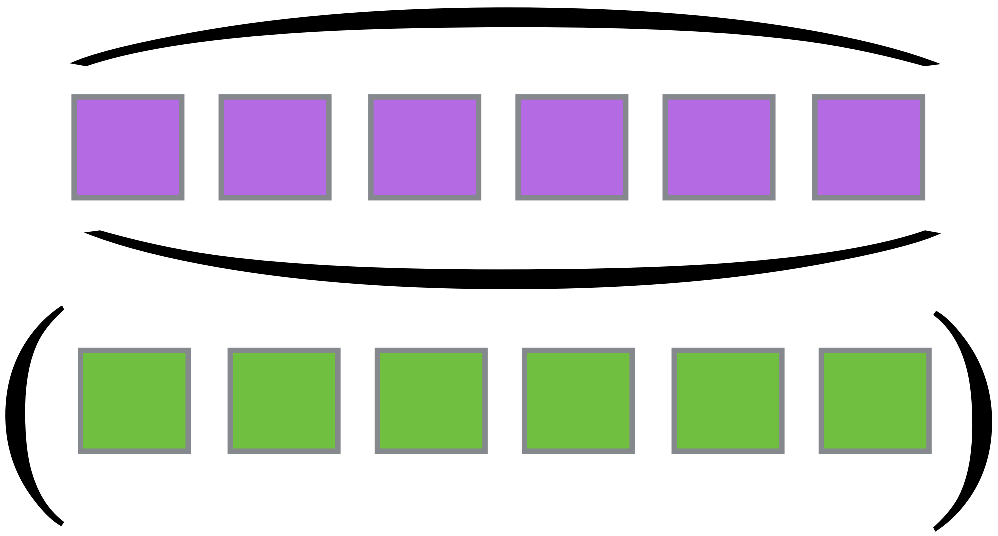
(row x column)



$$\mathbf{a}^T\mathbf{b} = \sum_{i=1}^{n} a_i b_i$$

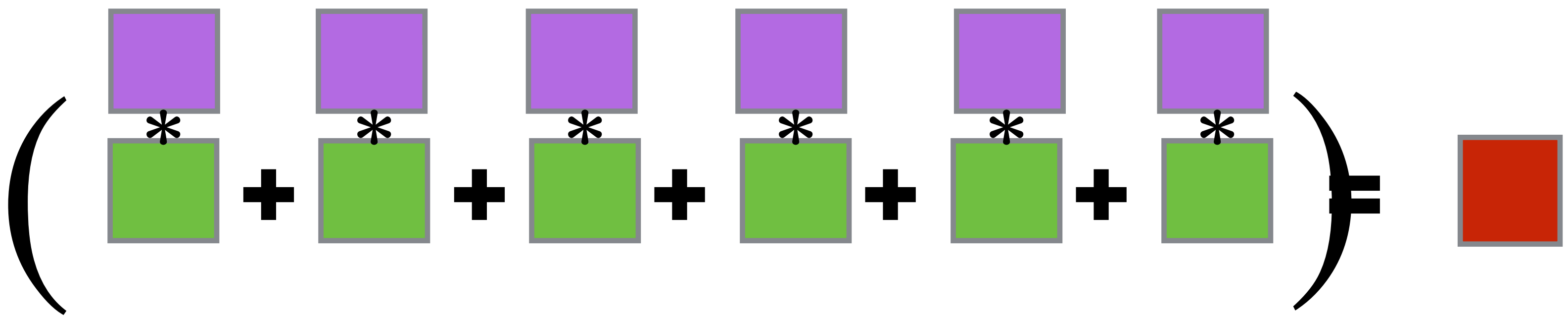
Inner Product

(row x column)

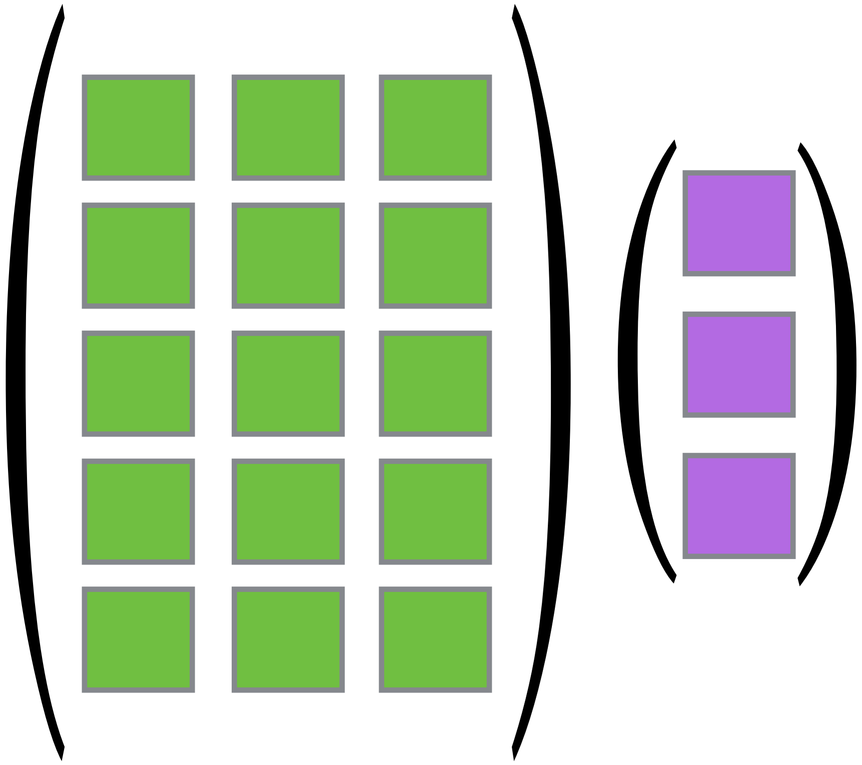


Inner Product

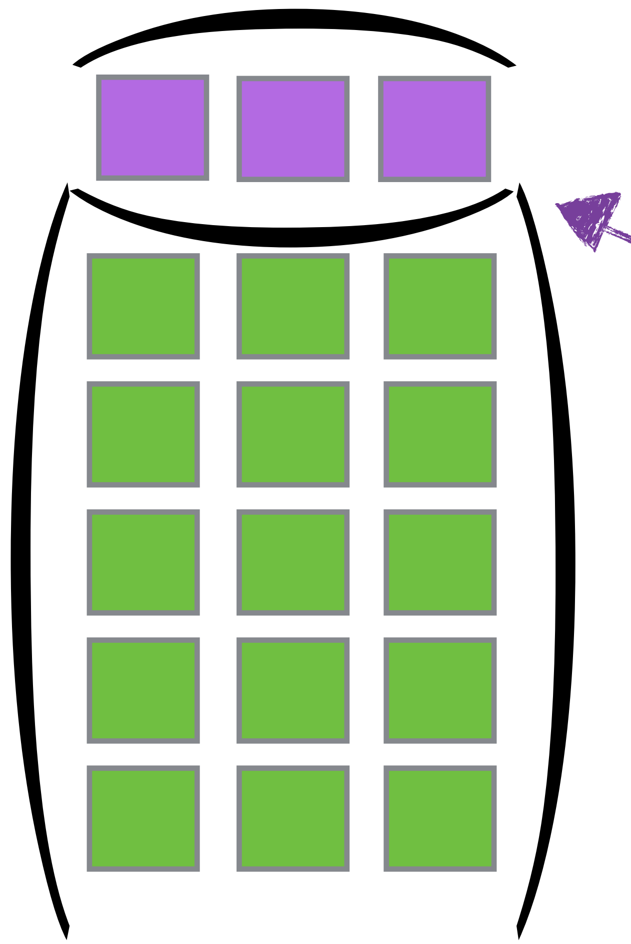
(row x column)



Matrix-Vector Multiplication (Inner-product view)

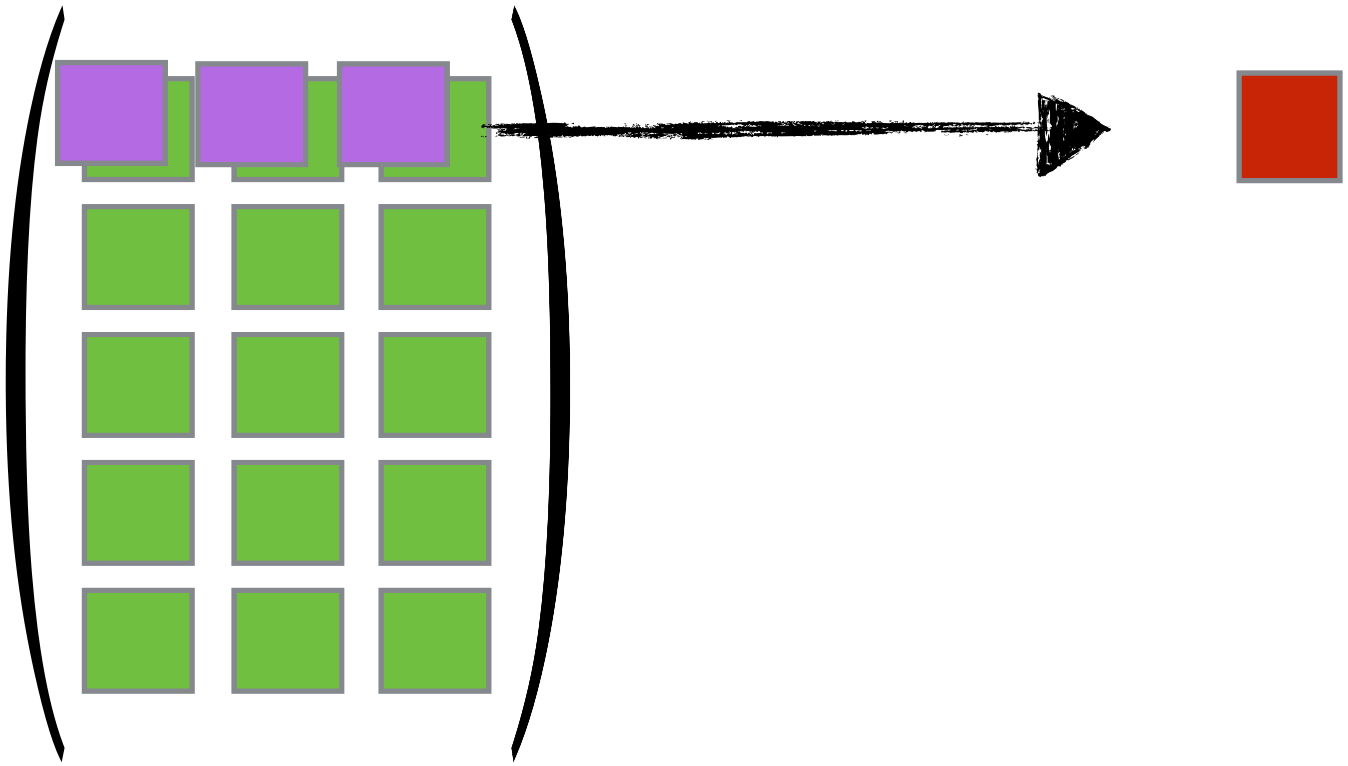


Matrix-Vector Multiplication (Inner-product view)

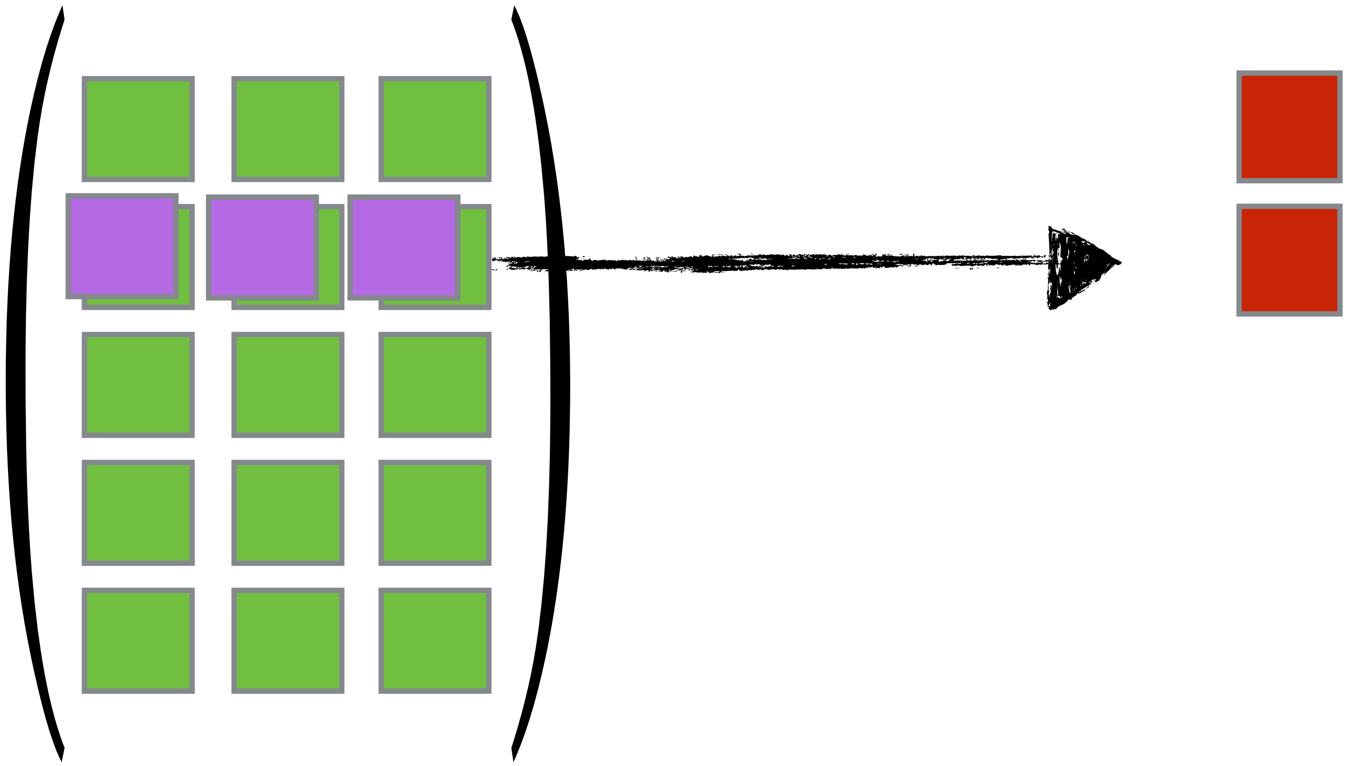


Sizes must match up!

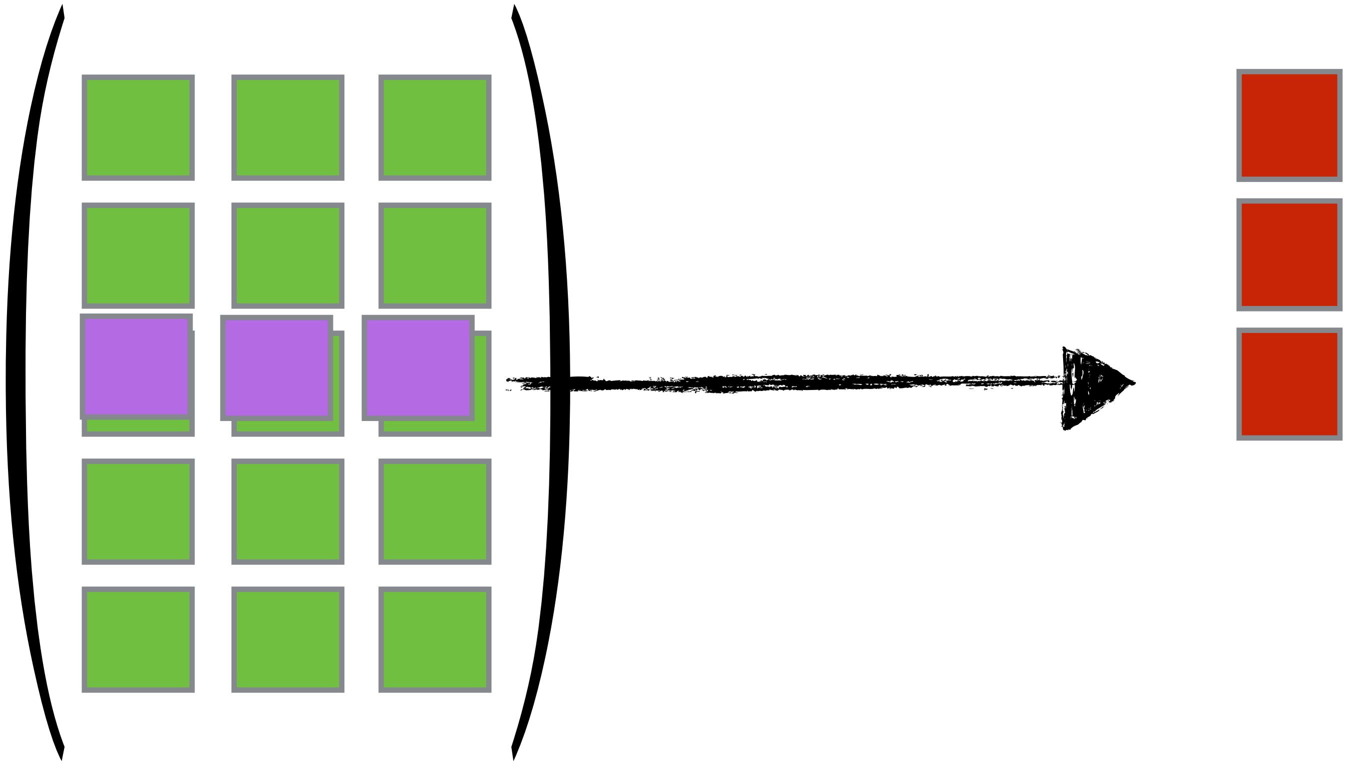
Matrix-Vector Multiplication (Inner-product view)



Matrix-Vector Multiplication (Inner-product view)



Matrix-Vector Multiplication (Inner-product view)



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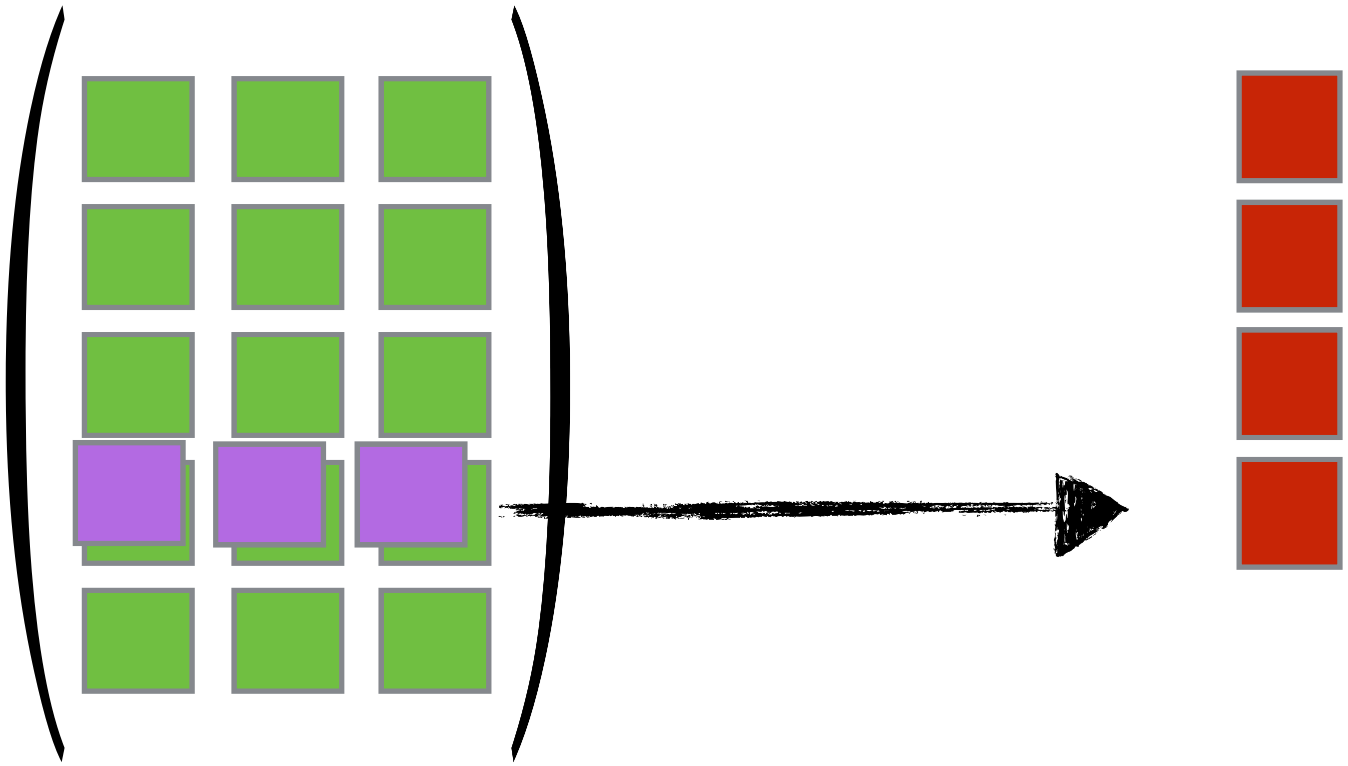
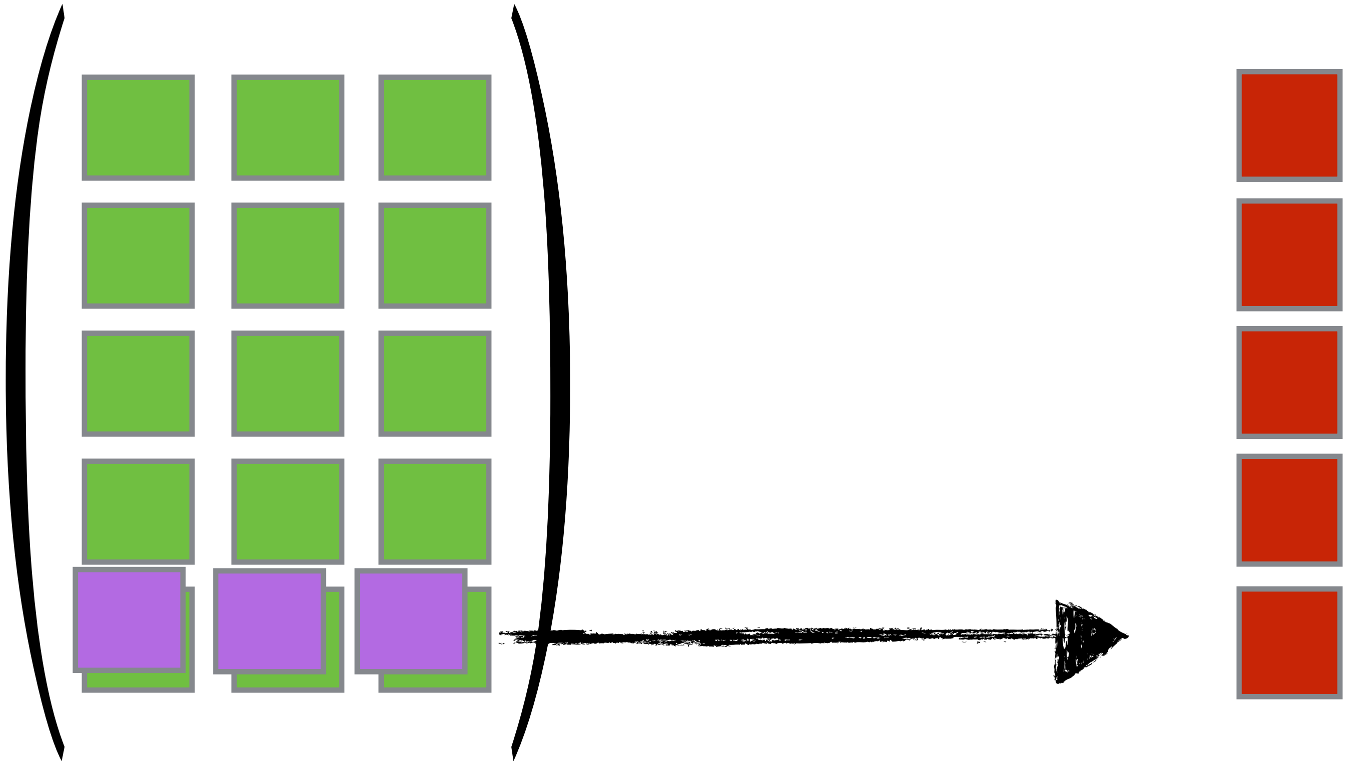


Matrix-Vector Multiplication (Inner-product view)



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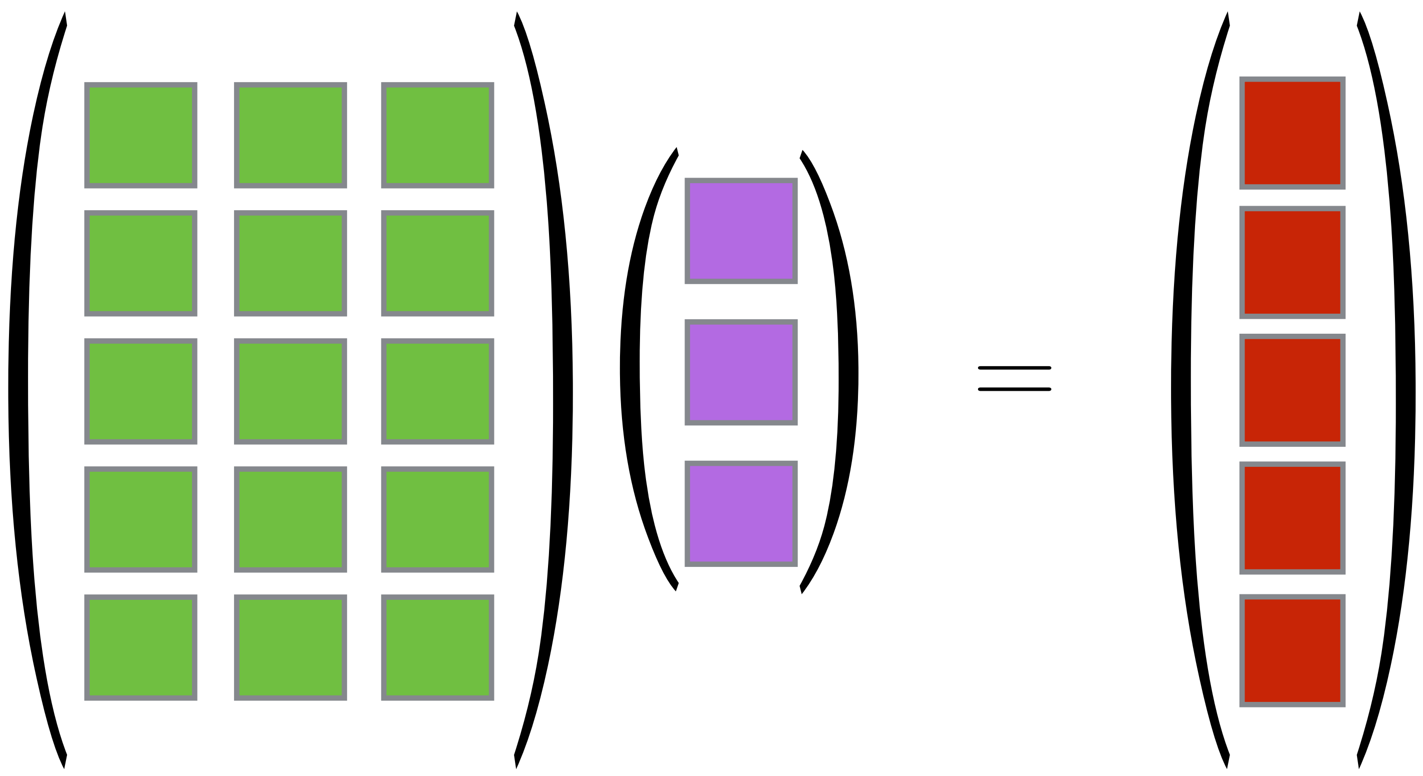
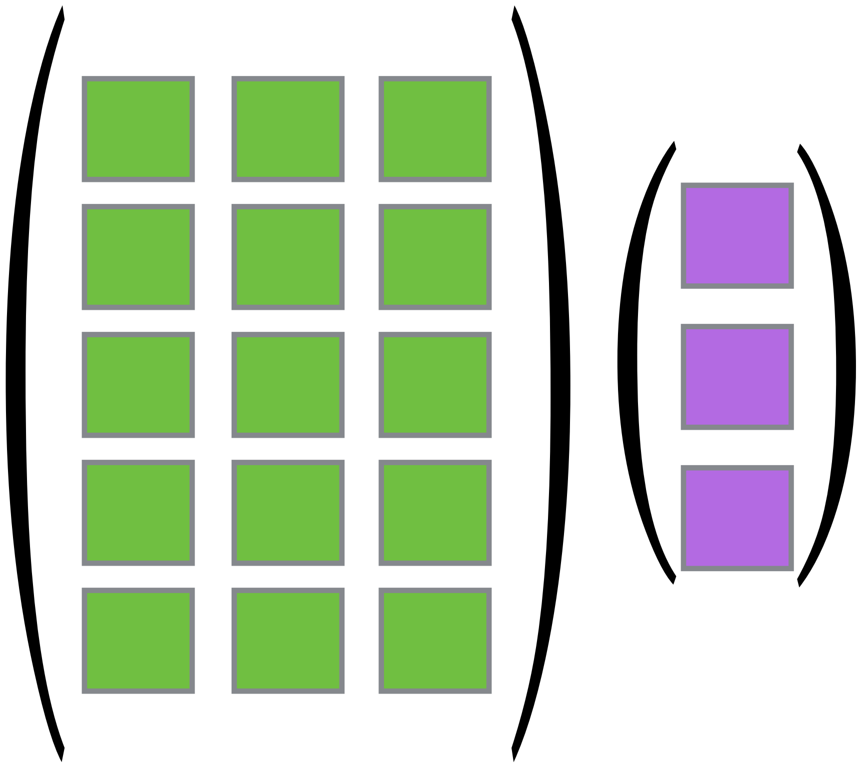


Matrix-Vector Multiplication (Linear Combination View)



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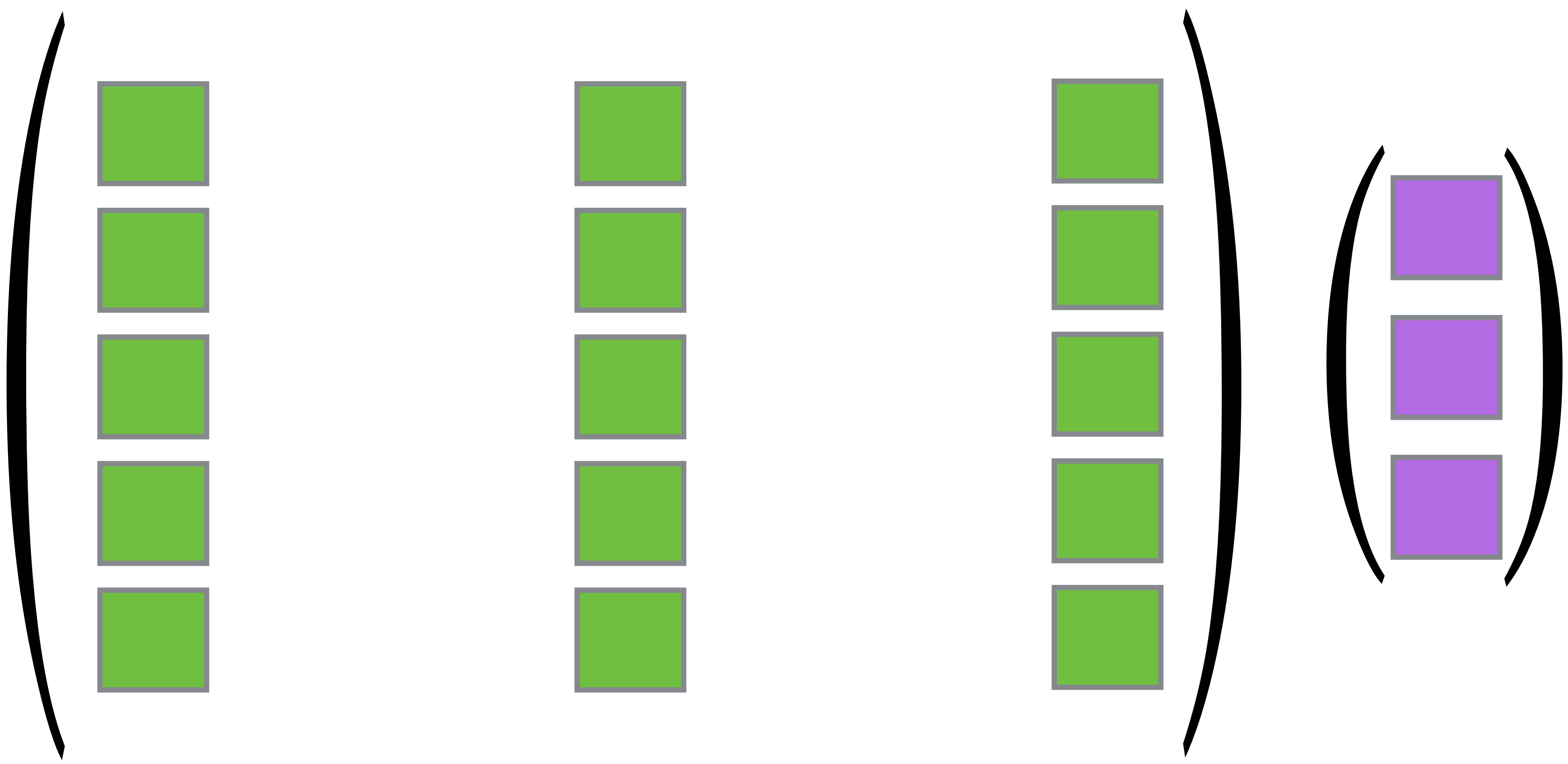

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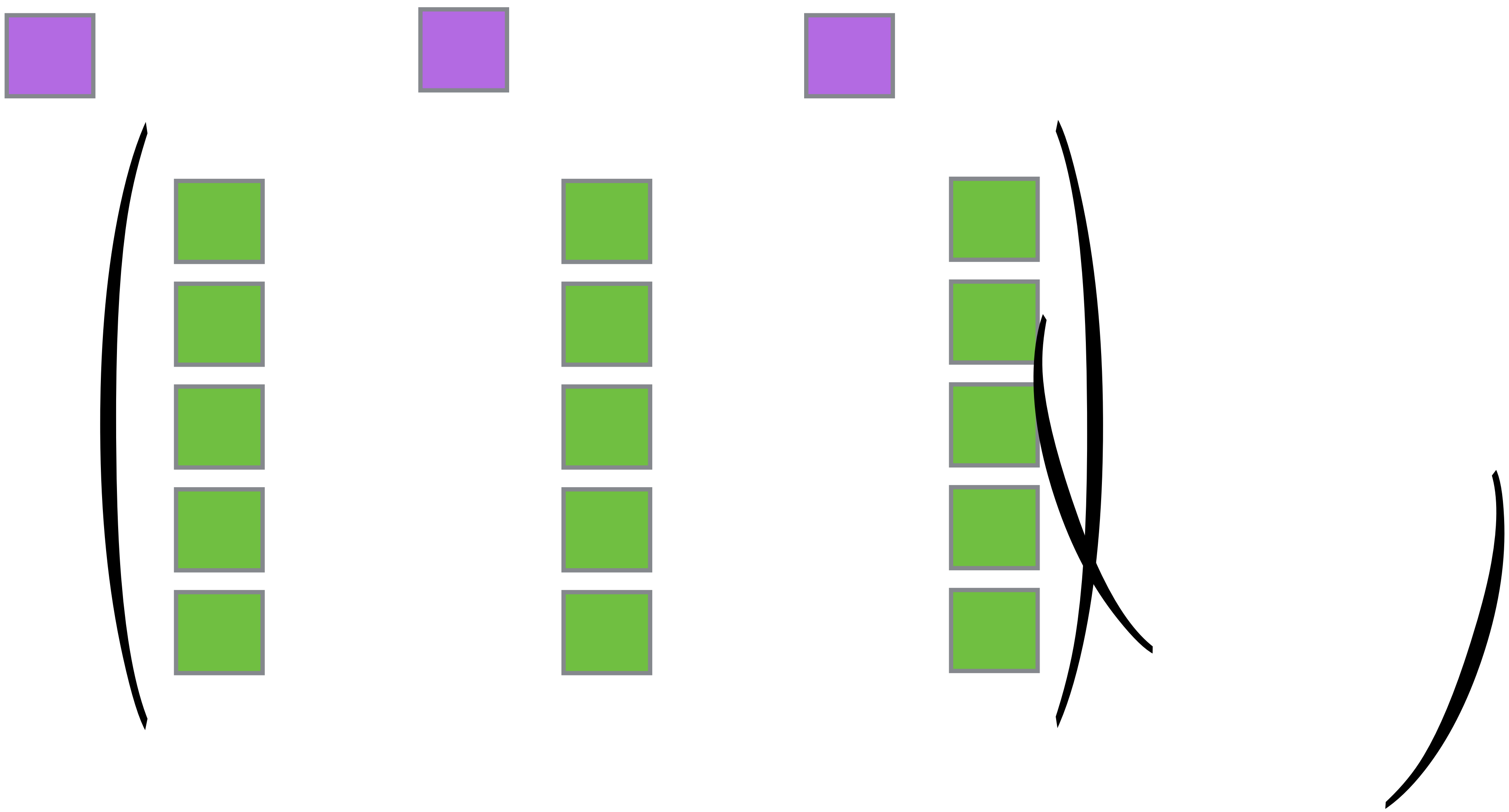

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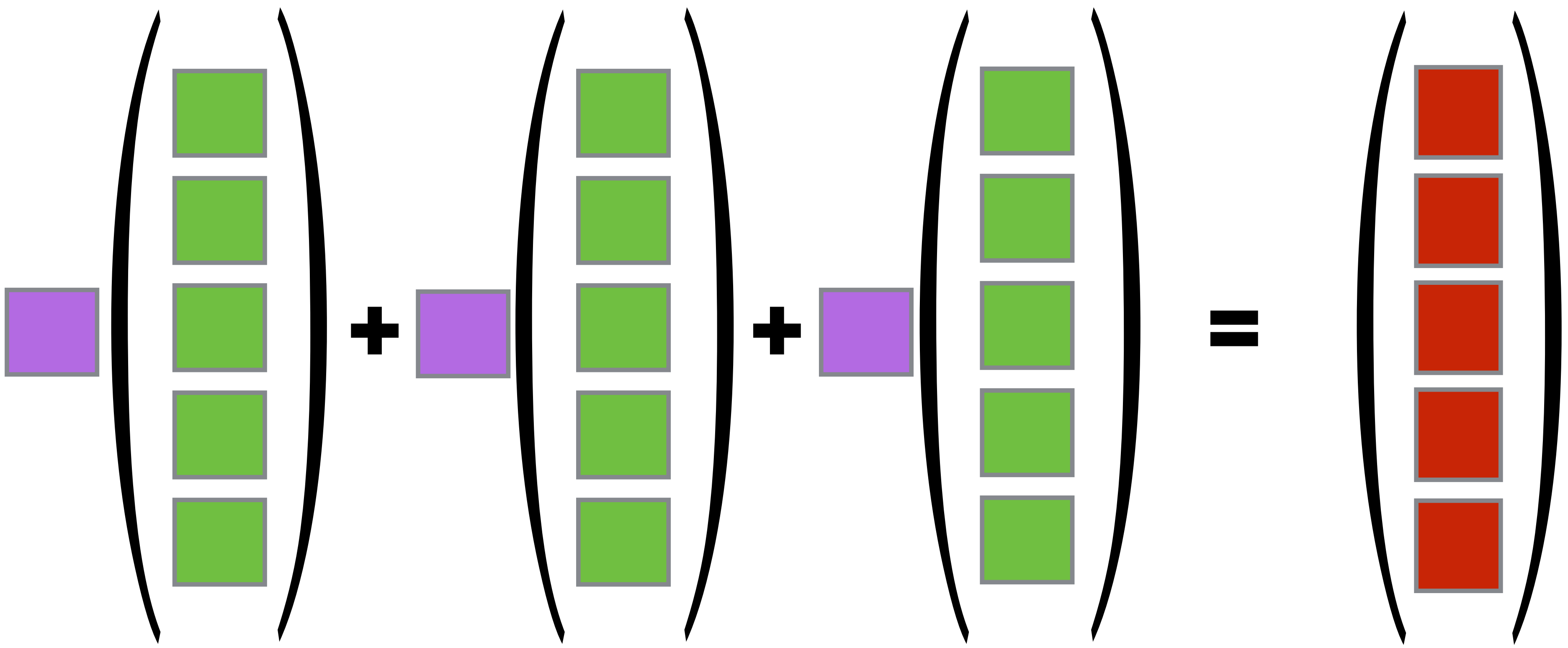
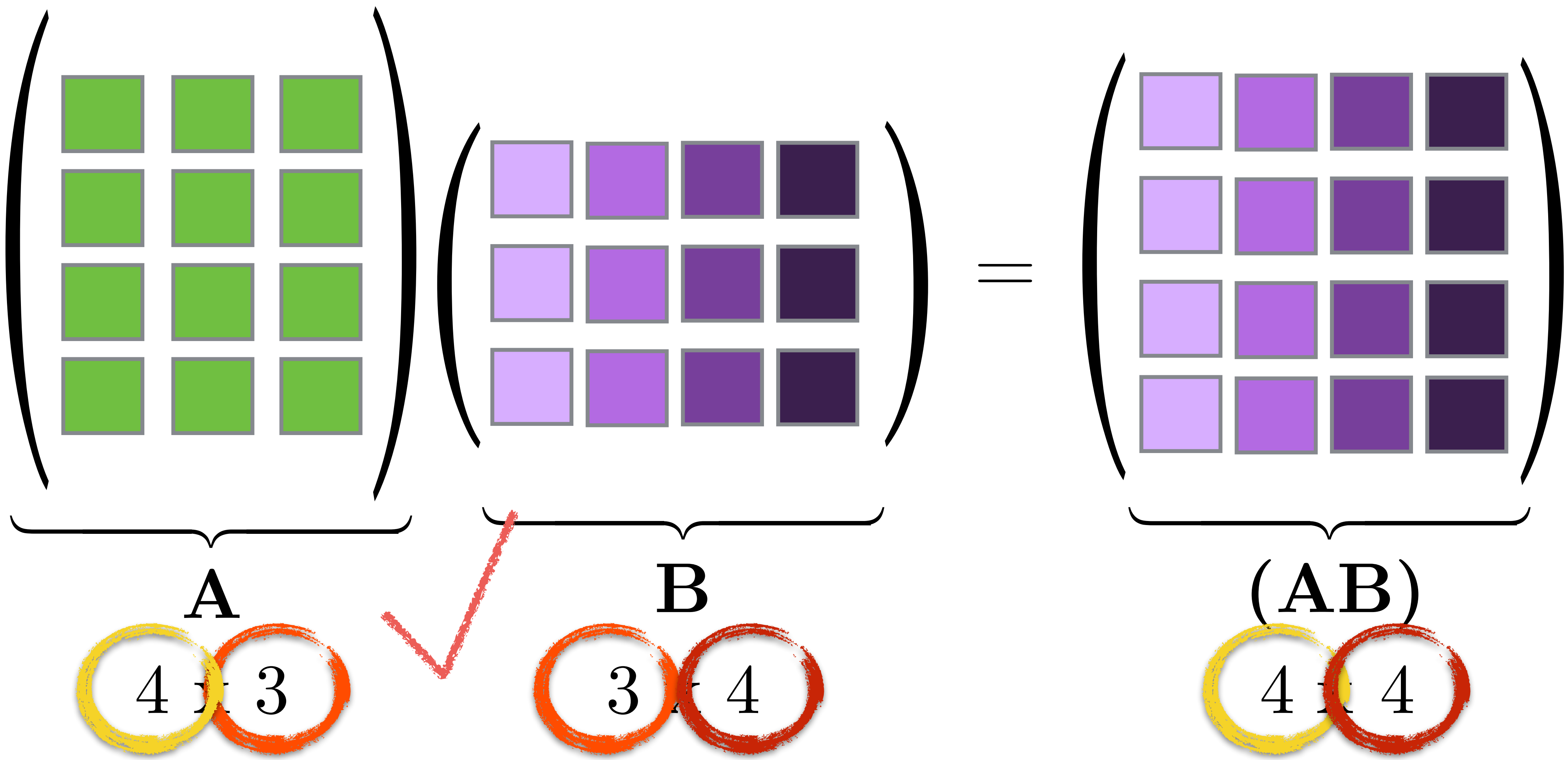


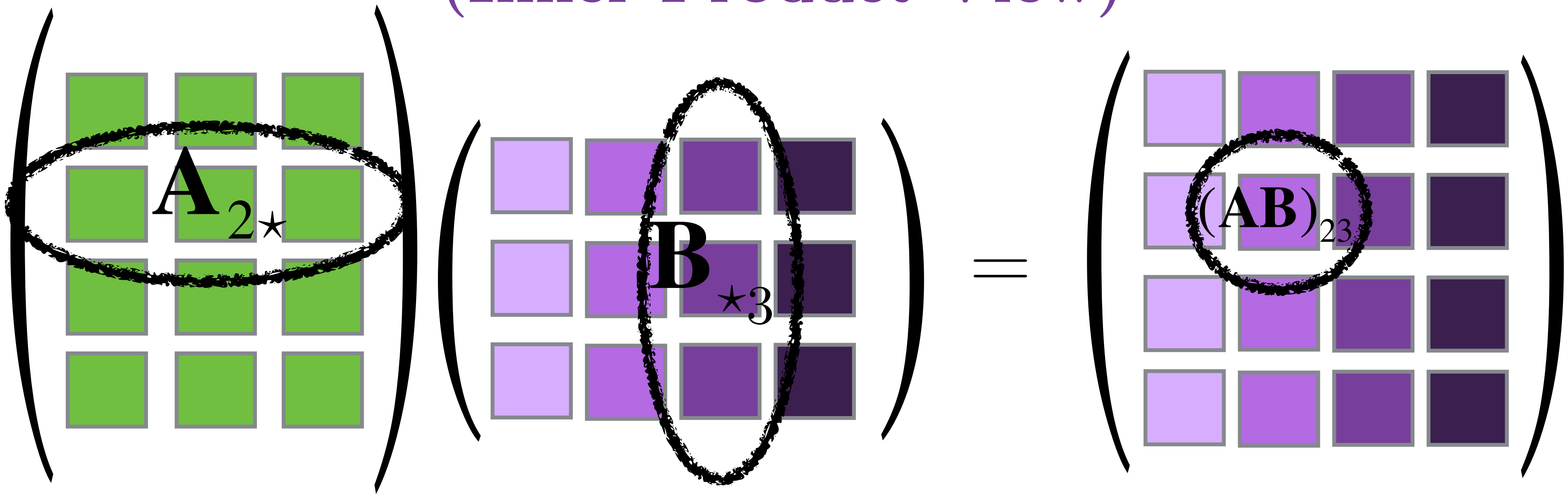
Matrix-Matrix Multiplication

Just a collection of matrix-vector products
(linear combinations) with different coefficients.



Matrix-Matrix Multiplication

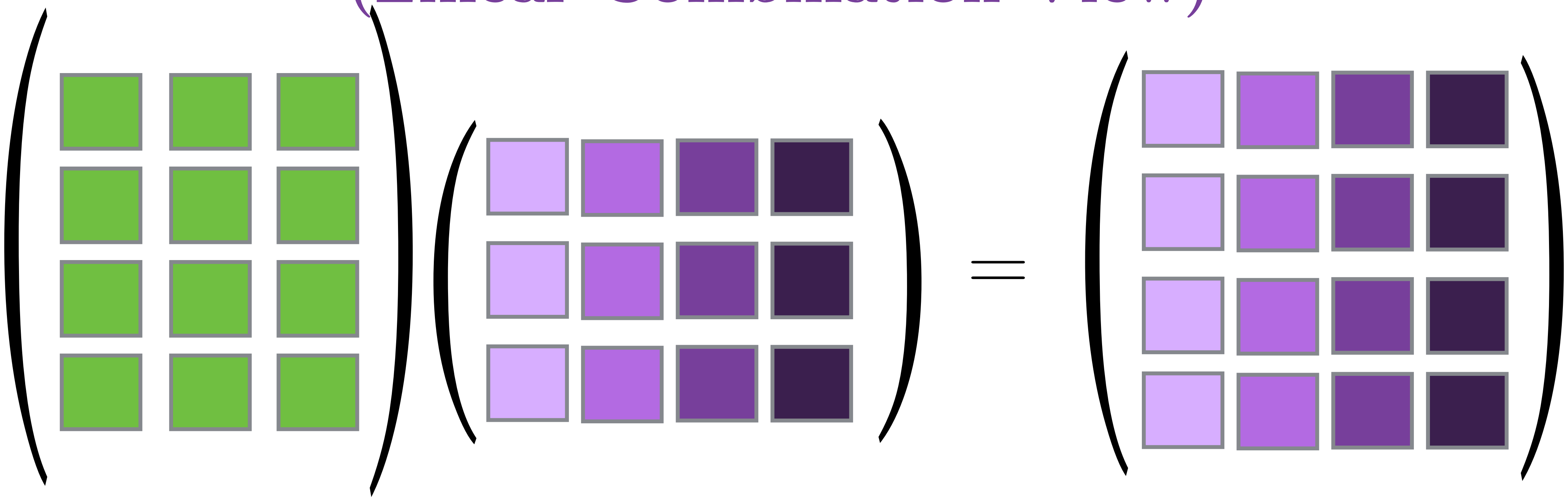
(Inner Product View)



$$(\mathbf{AB})_{ij} = \mathbf{A}_{i\star}\mathbf{B}_{\star j}$$

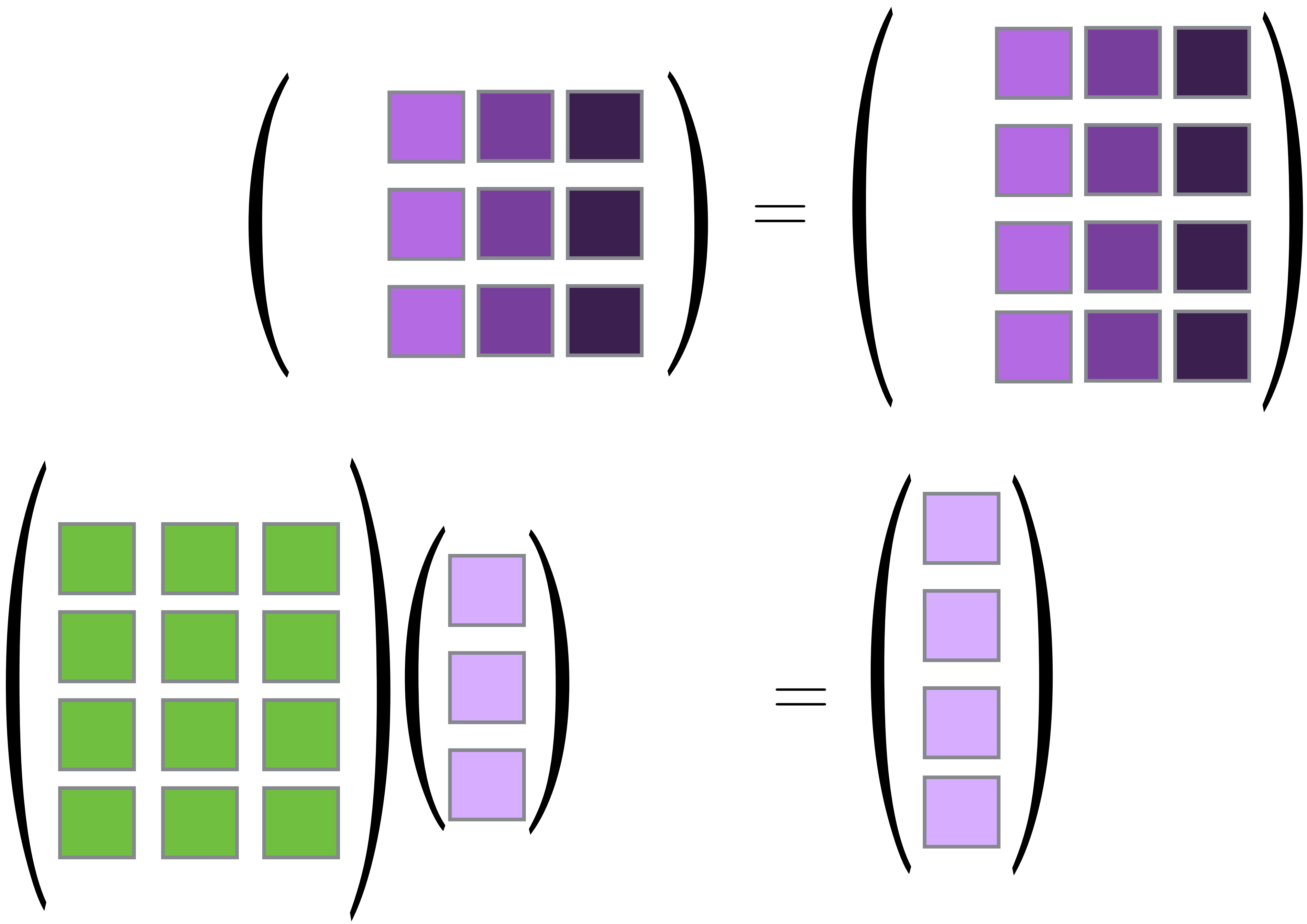
Matrix-Matrix Multiplication

(Linear Combination View)



Matrix-Matrix Multiplication

(Linear Combination View)



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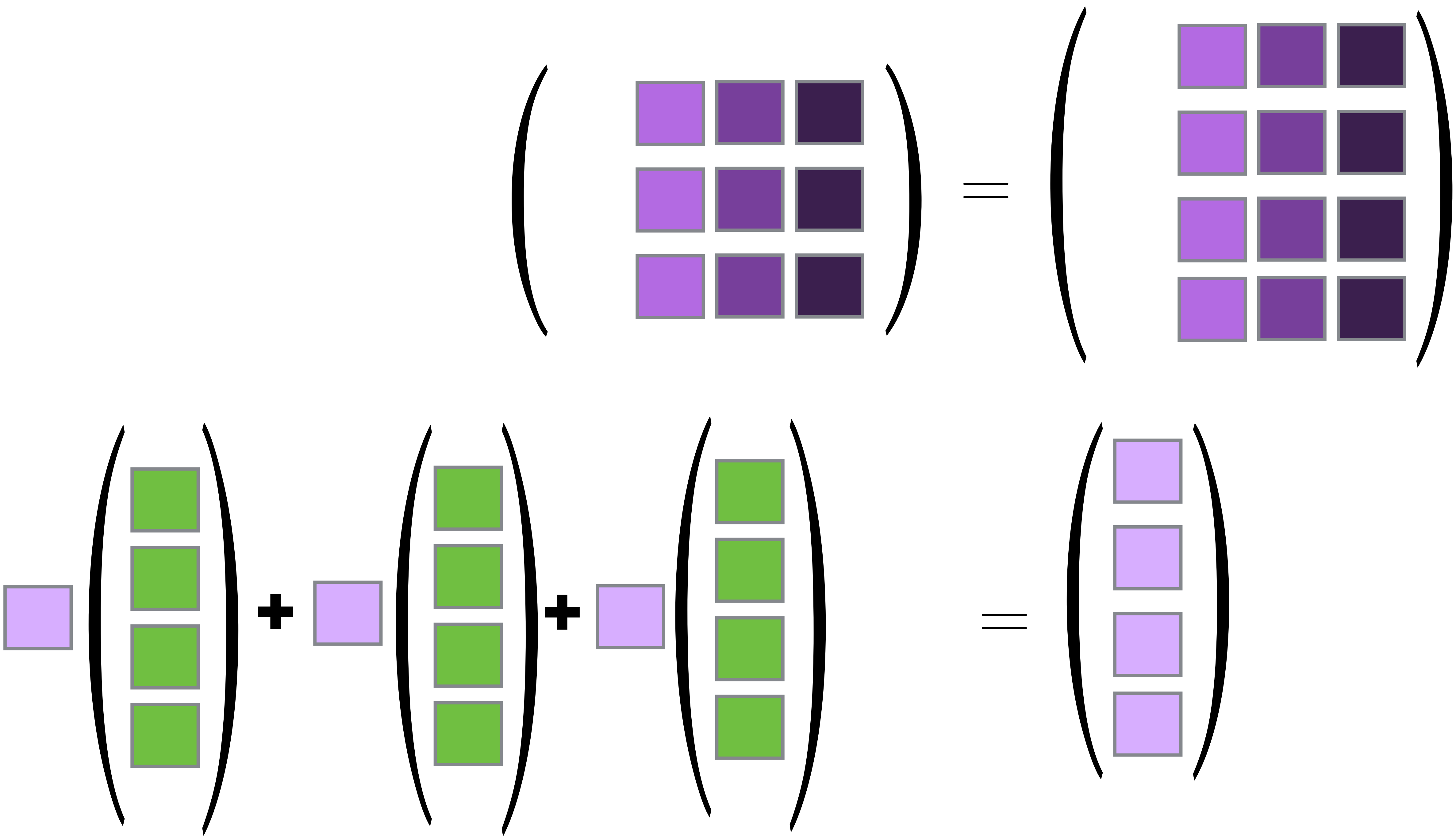

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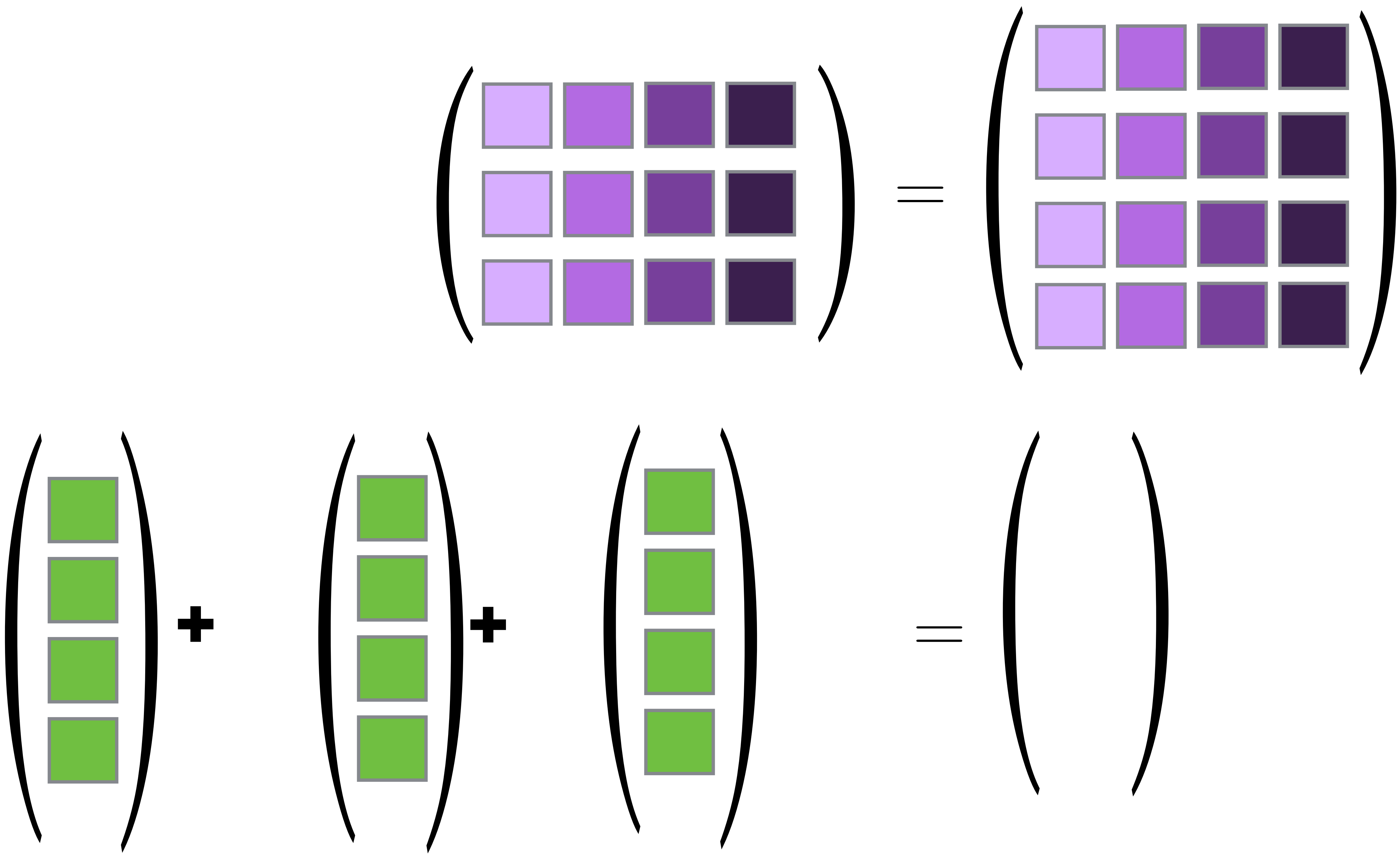

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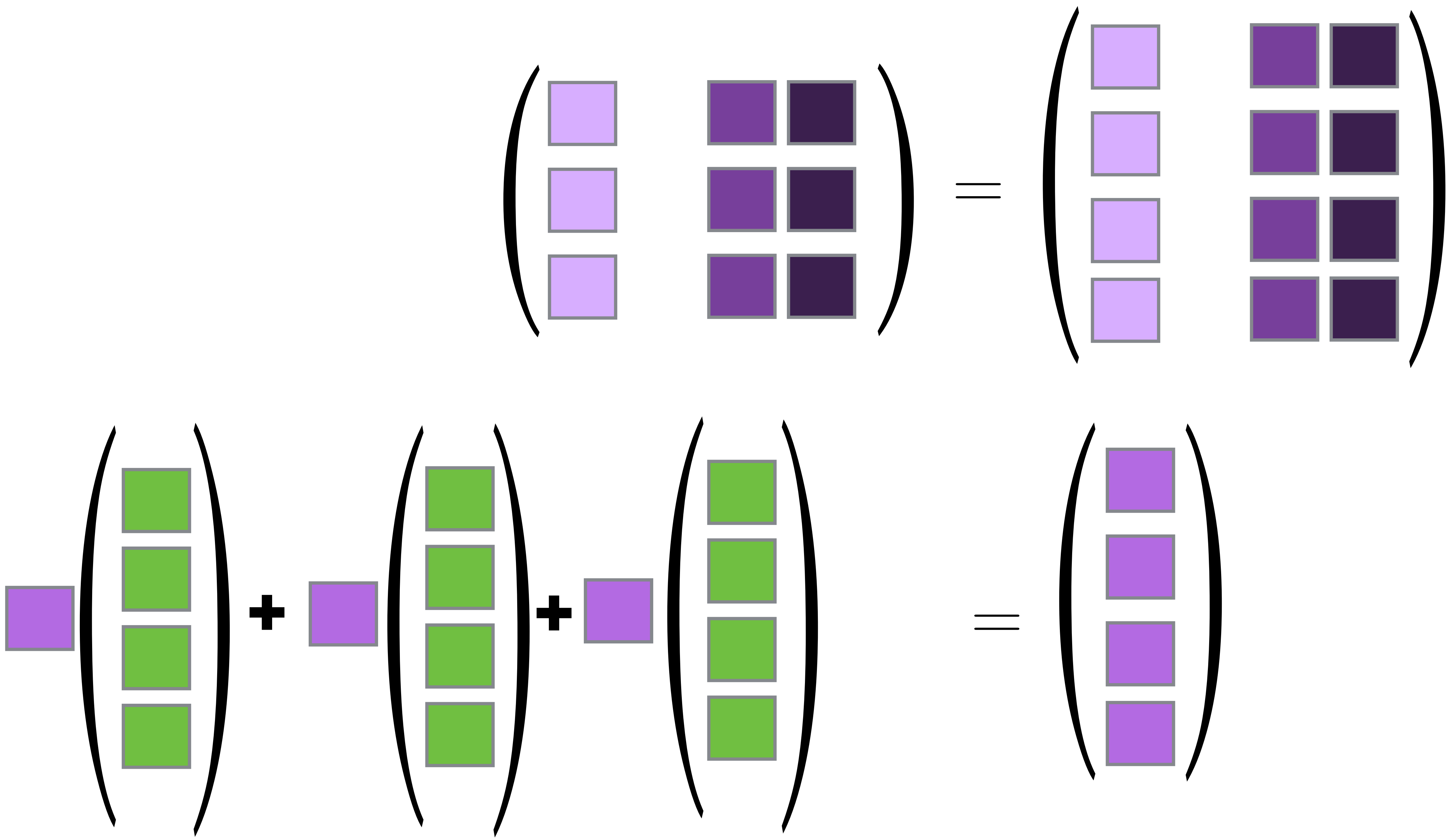

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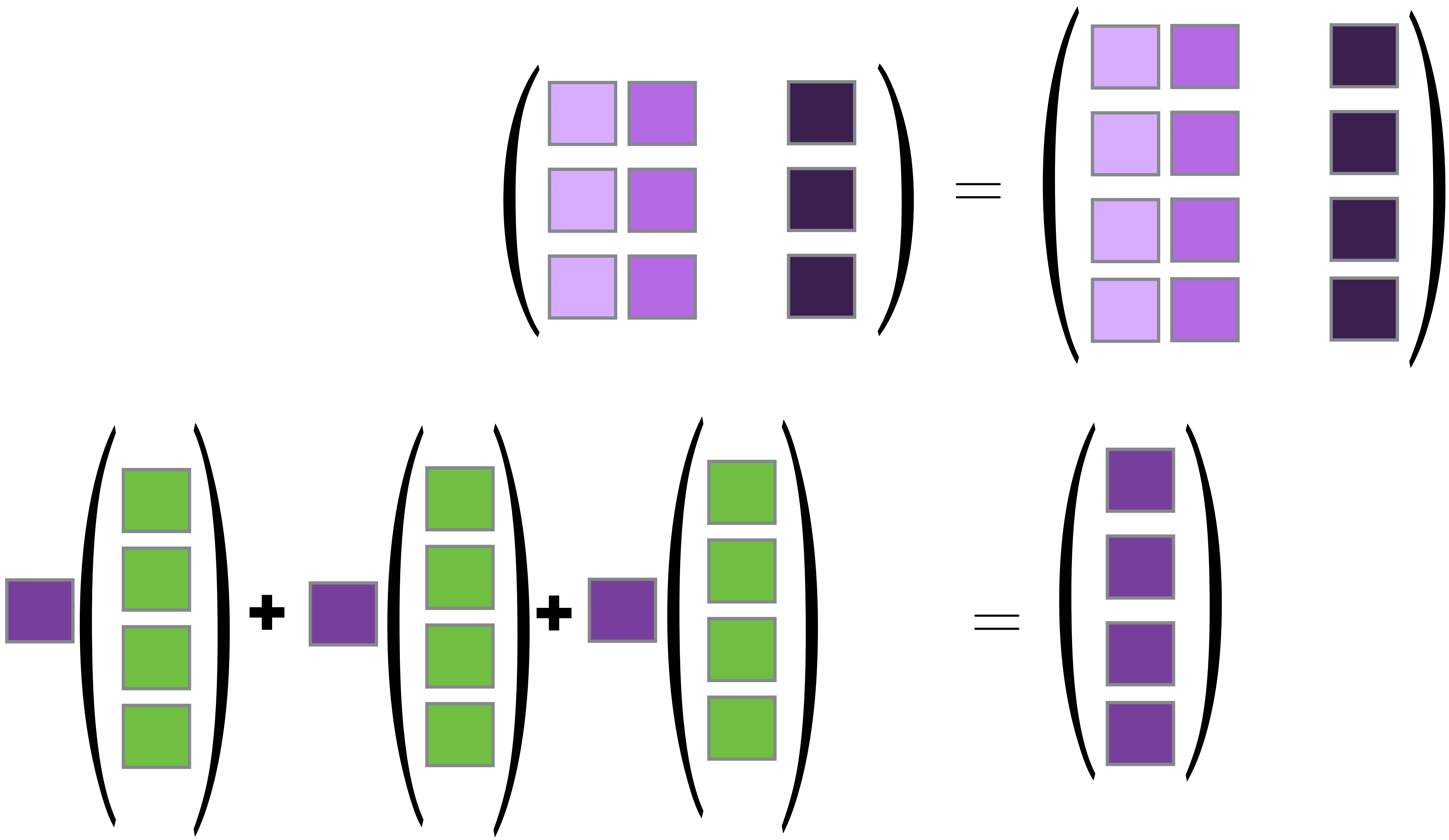
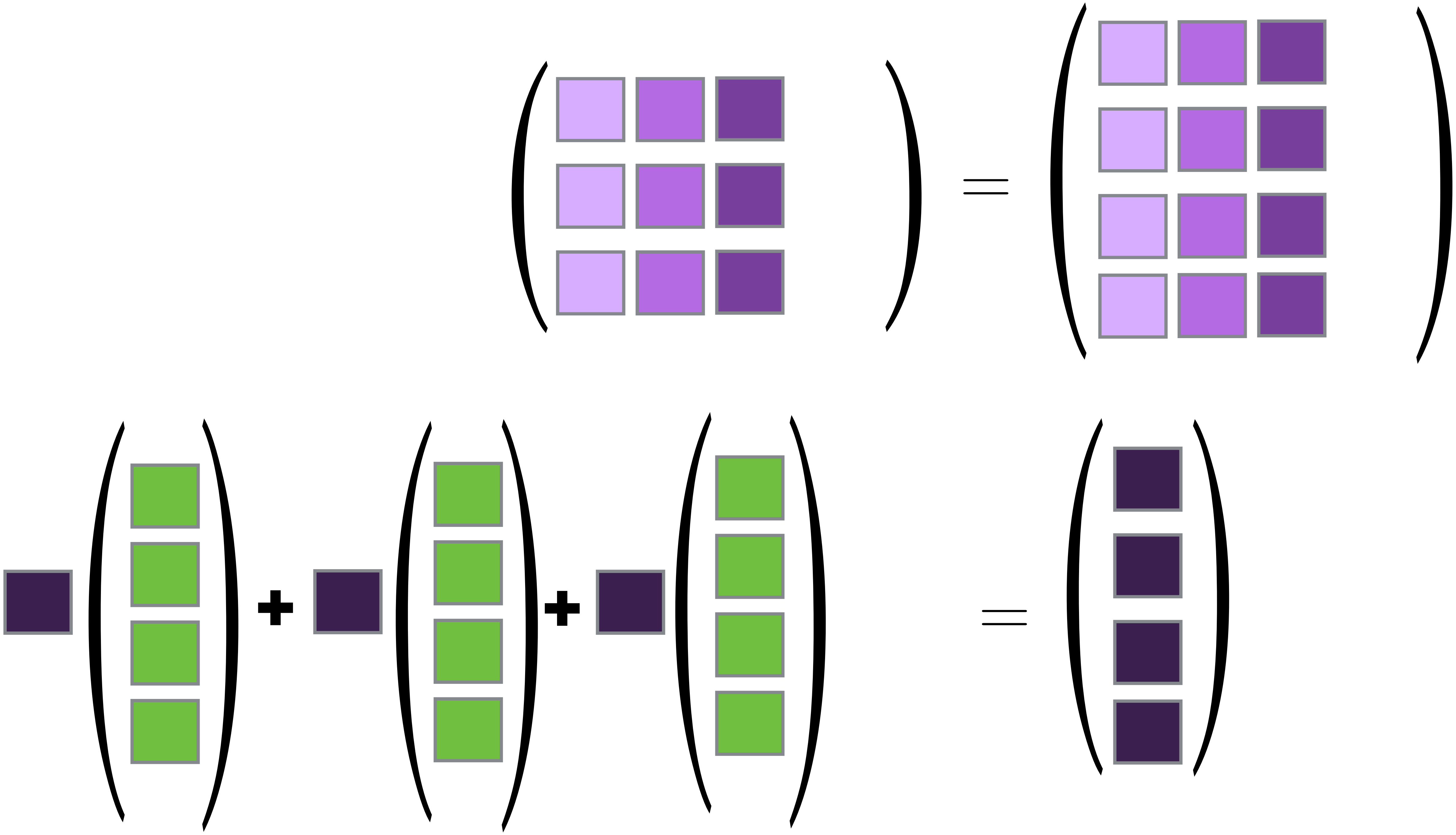


Matrix-Matrix Multiplication

(Linear Combination View)



Matrix-Matrix Multiplication

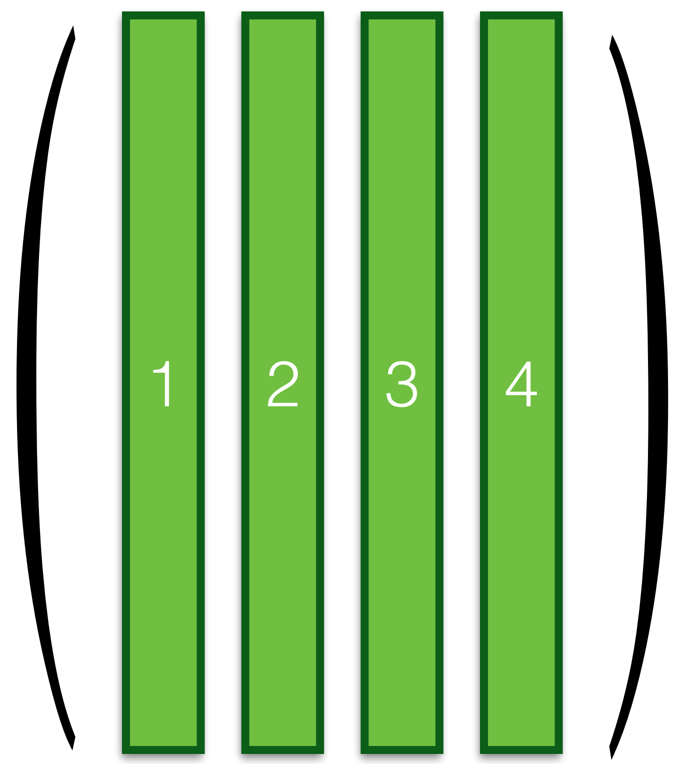
- MATRIX MULTIPLICATION IS NOT COMMUTATIVE! $\mathbf{AB} \neq \mathbf{BA}$
- Just a collection of matrix-vector products (linear combinations) with different coefficients.
- Each linear combination involves the same set of vectors (the green columns) with different coefficients (the purple columns).
- This has important implications!

More Matrix Operations and Special Matrices

Transpose Operator

The transpose of a matrix $\mathbf{A}$, written $\mathbf{A}^{\mathrm{T}}$ is the matrix whose rows are the columns of $\mathbf{A}$

$\mathbf{A} =$



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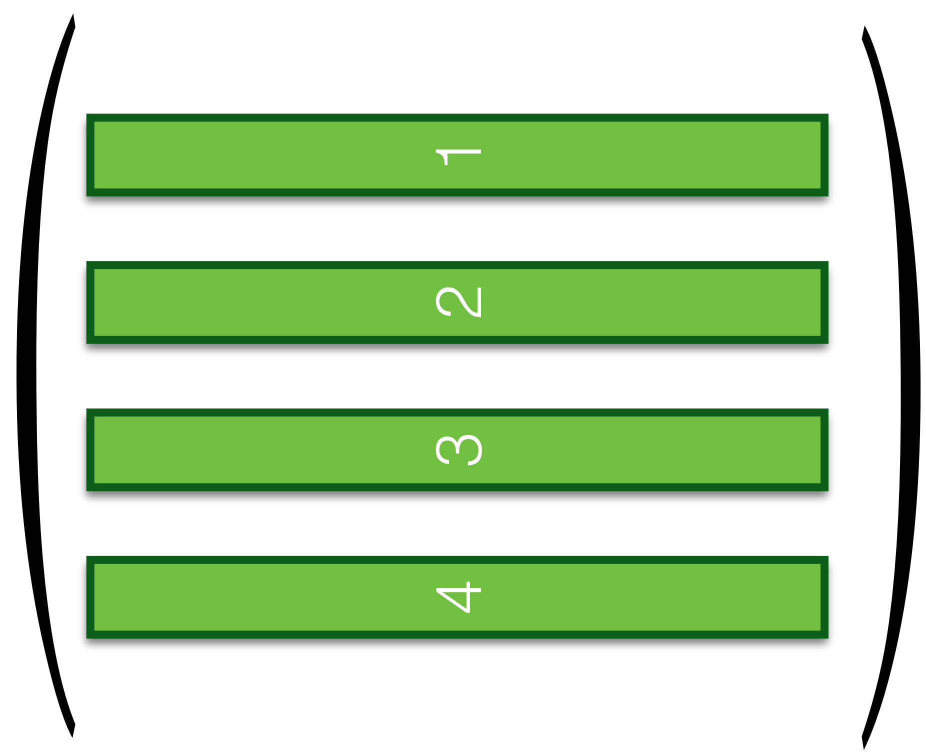

The transpose is useful for forming meaningful matrix products, typically of the form $\mathbf{A}^{\mathrm{T}}\mathbf{A}$.

The Identity Matrix

The **identity matrix**, denoted $\mathbf{I}$ is to matrix algebra what the number 1 is to scalar algebra. The multiplicative identity.

When multiplied by the identity, a matrix remains unchanged.

$$\mathbf{AI} = \mathbf{A}$$

$$\mathbf{IA} = \mathbf{A}$$

The Identity Matrix

The identity matrix is a matrix of zeros with 1's on the main diagonal.

$$\boldsymbol{I_1} = [\mathbf{1}],\ \boldsymbol{I_2} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix},\ \boldsymbol{I_3} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix},\ \cdots,\ \boldsymbol{I_n} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \end{bmatrix}.$$

The Inverse Matrix

The **inverse** of a matrix $\mathbf{A}$, should it exist, is denoted $\mathbf{A}^{-1}$, is a matrix for which multiplication by $\mathbf{A}$ results in the identity matrix.

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The Inverse Matrix

All operations involving "cancelling" terms must be done with an inverse matrix.

$$\mathbf{A}\cancel{\mathbf{x}} = \lambda\cancel{\mathbf{x}} \quad ?$$

No.

Systems of Equations

Systems of Equations

$$\begin{cases} \quad\quad 2x_2 + 3x_3 = 8 \\ 2x_1 + 3x_2 + 1x_3 = 5 \\ x_1 - x_2 - 2x_3 = -5 \end{cases}$$

$$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ -5 \end{pmatrix}$$

Systems of Equations
(Three types)

- In some applications, systems of equations have an **exact solution** - but this is rare.
- The system of equations may be a set of constraints ($\leq$, $=$, $\geq$). **Infinitely many solutions** within the constraints and must optimize some other quantity.
- In most applications, there is **no exact solution**. We introduce an error term and try to minimize it.

Systems of Equations

(Least Squares)

Obs	Weight	Width	Length	Time
1	3	5.4	6.3	10.11
2	1.1	1.2	2.1	4.25
3	2.4	3.4	5	8.09
4	1.9	2.8	8.1	7.20
5	3.2	6.1	4.5	9.90
6	2.7	3.7	4.6	7.75

$$\textbf{Time} = \beta_0 + \beta_1\text{Weight} + \beta_2\text{Width} + \beta_3\text{Length}$$

Systems of Equations

(Least Squares)

Obs	Weight	Width	Length	Time
1	**3**	**5.4**	**6.3**	**10.11**
2	1.1	1.2	2.1	4.25
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4	1.9	2.8	8.1	7.20
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6	2.7	3.7	4.6	7.75

$$\textbf{Time} = \beta_0 + \beta_1 \text{Weight} + \beta_2 \text{Width} + \beta_3 \text{Length}$$

$$\mathbf{10.11} = 1\beta_0 + \mathbf{3}\beta_1 + \mathbf{5.4}\beta_2 + \mathbf{6.3}\beta_3$$

$$\mathbf{8.09} = 1\beta_0 + \mathbf{2.4}\beta_1 + \mathbf{3.4}\beta_2 + \mathbf{5}\beta_3$$

Systems of Equations

(Least Squares)

$$\begin{array}{ccccc} \underline{\text{Intercept}} & \underline{\text{Weight}} & \underline{\text{Width}} & \underline{\text{Length}} & \underline{\text{Time}} \end{array}$$

$$\beta_0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \beta_1 \begin{pmatrix} 3 \\ 1.1 \\ 2.4 \\ 1.9 \\ 3.2 \\ 2.7 \end{pmatrix} + \beta_2 \begin{pmatrix} 5.4 \\ 1.2 \\ 3.4 \\ 2.8 \\ 6.1 \\ 3.7 \end{pmatrix} + \beta_3 \begin{pmatrix} 6.3 \\ 2.1 \\ 5 \\ 8.1 \\ 4.5 \\ 4.6 \end{pmatrix} \approx \begin{pmatrix} 10.11 \\ 4.25 \\ 8.09 \\ 7.20 \\ 9.90 \\ 7.75 \end{pmatrix}$$

$$\textbf{Time} \cancel{=} \hat{\beta}_0 + \hat{\beta}_1 \text{Weight} + \hat{\beta}_2 \text{Width} + \hat{\beta}_3 \text{Length} + \varepsilon$$

Systems of Equations
(Least Squares)

Intercept	Weight	Width	Length	Time
1	3	5.4	6.3	10.11
1	1.1	1.2	2.1	4.25
1	2.4	3.4	5	8.09
1	1.9	2.8	8.1	7.20
1	3.2	6.1	4.5	9.90
1	2.7	3.7	4.6	7.75

$$\underbrace{\begin{pmatrix} 1 & 3 & 5.4 & 6.3 \\ 1 & 1.1 & 1.2 & 2.1 \\ 1 & 2.4 & 3.4 & 5 \\ 1 & 1.9 & 2.8 & 8.1 \\ 1 & 3.2 & 6.1 & 4.5 \\ 1 & 2.7 & 3.7 & 4.6 \end{pmatrix}}_{\mathbf{X}} \qquad \underbrace{\begin{pmatrix} 10.11 \\ 4.25 \\ 8.09 \\ 7.20 \\ 9.90 \\ 7.75 \end{pmatrix}}_{\mathbf{y}}$$

$$\textbf{Time} = \hat{\beta}_0 + \hat{\beta}_1 \text{Weight} + \hat{\beta}_2 \text{Width} + \hat{\beta}_3 \text{Length} + \varepsilon$$

$$\mathbf{y} = \mathbf{X}\hat{\boldsymbol{\beta}} + \varepsilon$$

Systems of Equations

(Least Squares)

$$\boxed{\mathbf{y} = \mathbf{X}\boldsymbol{\beta}}$$ (has no solutions. "inconsistent")

Want to find $\boldsymbol{\beta}$ that gets the modeled values ($\hat{\mathbf{y}}=\mathbf{X}\boldsymbol{\beta}$) on the right as close as possible to the true values ($\mathbf{y}$) on the left.

Minimize squared error

$$\min_{\beta} \sum_{i} \varepsilon_i^{\,2}$$

$$\boxed{\varepsilon = \mathrm{y} - \mathbf{X}\boldsymbol{\beta}}$$

Systems of Equations

(Least Squares)

Minimize squared error

$$\min_{\beta} \sum_{i} \varepsilon_i^{\,2}$$

$$\varepsilon = y - \mathbf{X}\beta$$

or equivalently

$$\min_{\beta} (\mathbf{y} - \mathbf{X}\beta)^T (\mathbf{y} - \mathbf{X}\beta)$$

or equivalently

$$\min_{\beta} ||(\mathbf{y} - \mathbf{X}\beta)||_2^2$$

Systems of Equations
(Least Squares)

HOW to find the least squares solution?

<u>The Normal Equations</u>

$$\mathbf{X}^{\mathbf{T}}\mathbf{X}\boldsymbol{\beta} = \mathbf{X}^{\mathbf{T}}\mathbf{y}$$

As long as X is full rank (no perfect multicollinearity), $\mathbf{X}^{\mathbf{T}}\mathbf{X}$ has an inverse and this system has an exact solution.

That solution **IS** the least squares solution.

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^{\mathbf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathbf{T}}\mathbf{y}$$

We're DONE talking about regression in Linear Algebra class.

From now on, our focus is on *unsupervised* problems that do not have a target variable.

Norms, Distances, and Similarity

Norms

- **Norms** are functions that measure the *magnitude* or *length* of a vector.
- Written $||\mathbf{x}||$
- 2-Norm (Euclidean norm) is the most common.

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\mathbf{x}^T\mathbf{x}}$$

Norms

- The distance between two points, **x** and **y**, is the norm of their difference.

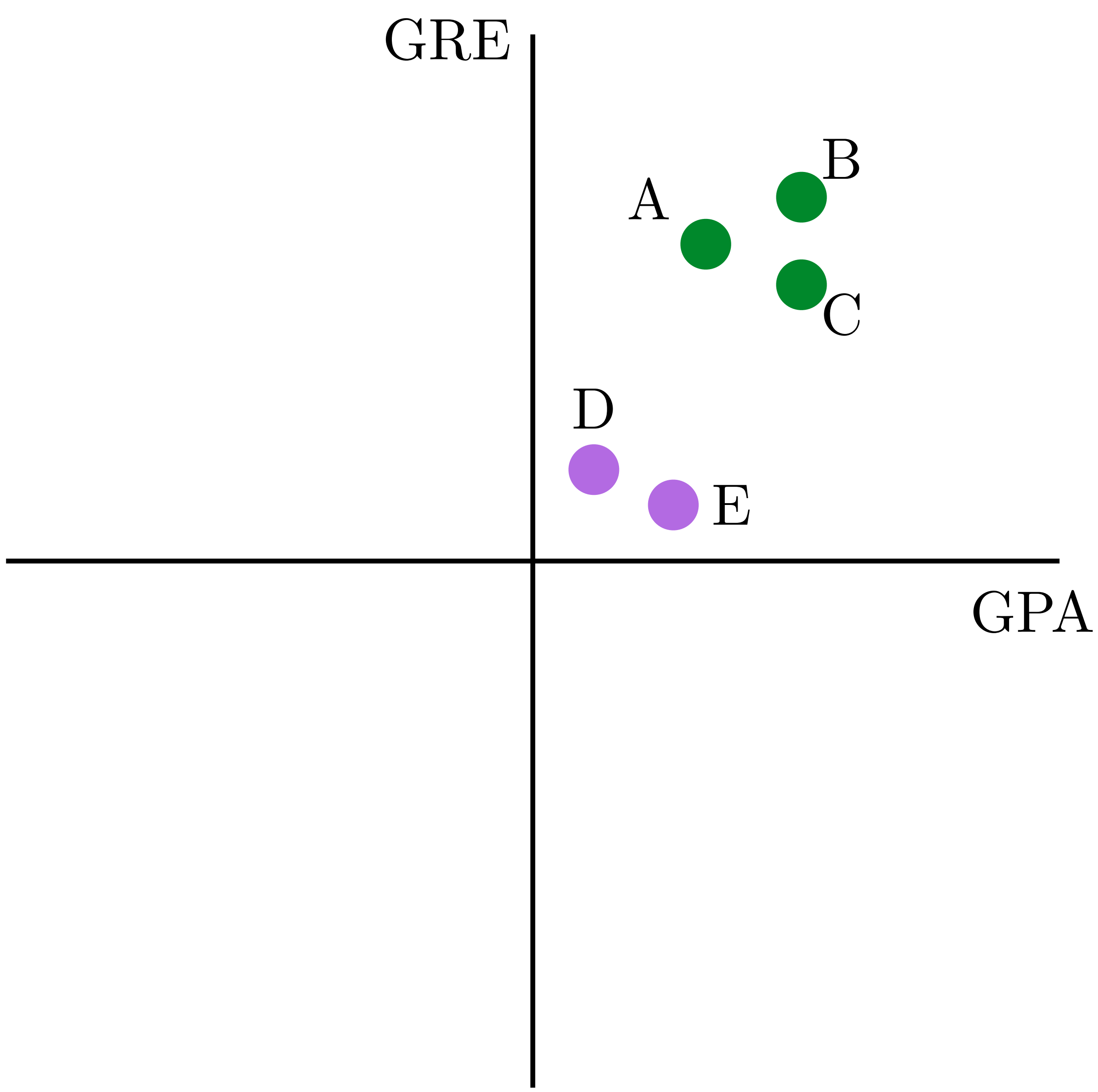
$$\|\mathbf{x}-\mathbf{y}\|$$

- We can use this information to determine which points are more similar to each other.

- May create a **distance matrix**, **D**, which contains pairwise distances between points (observations).

$$\mathbf{D}_{ij} = \| obs_i - obs_j \|$$

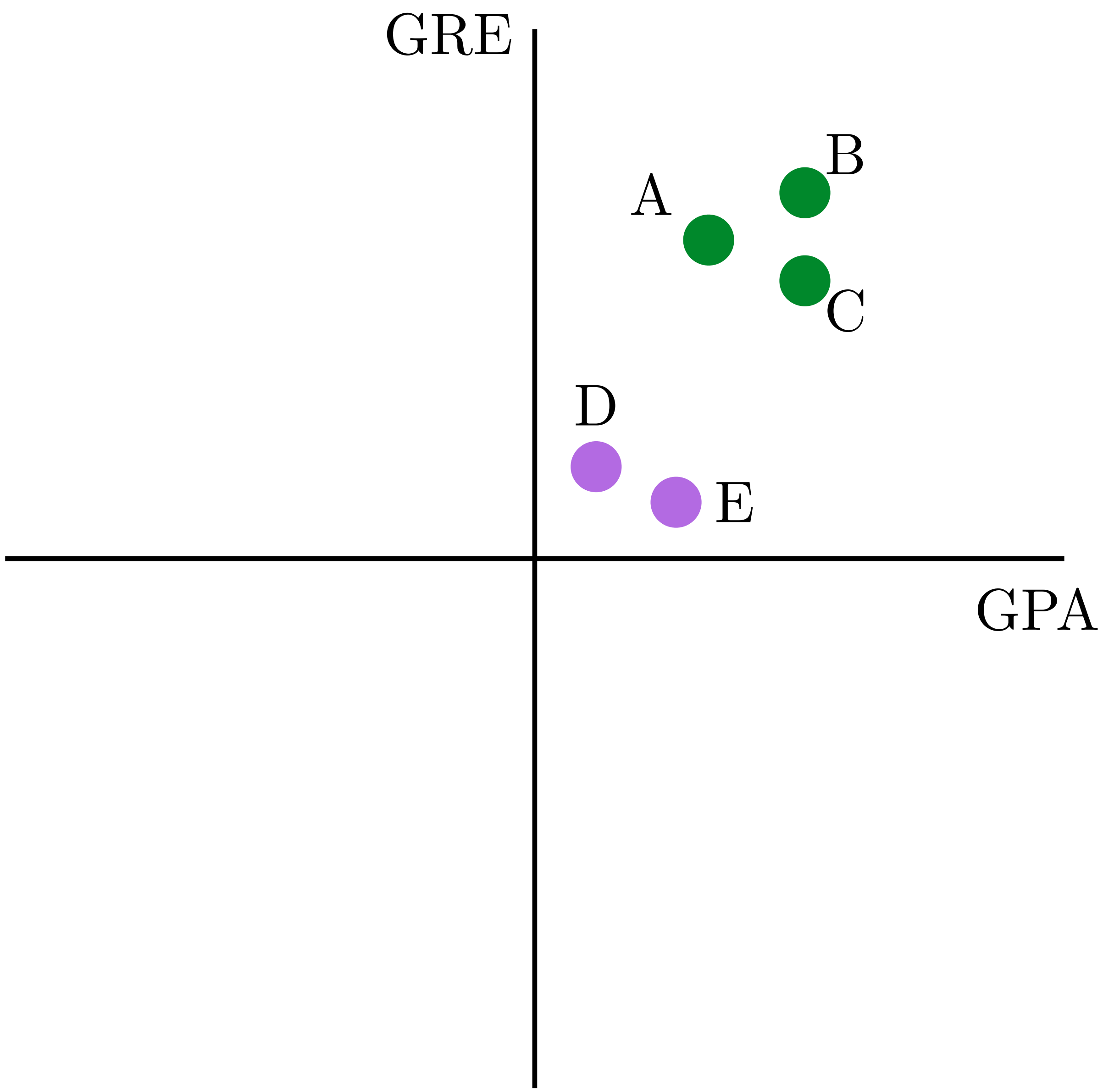
Distance Matrix



$\mathbf{D} =$

	A	B	C	D	E
A	0	1	1	3	4
B	1	0	1	5	5
C	1	1	0	4	3
D	3	5	4	0	1
E	4	5	3	1	0

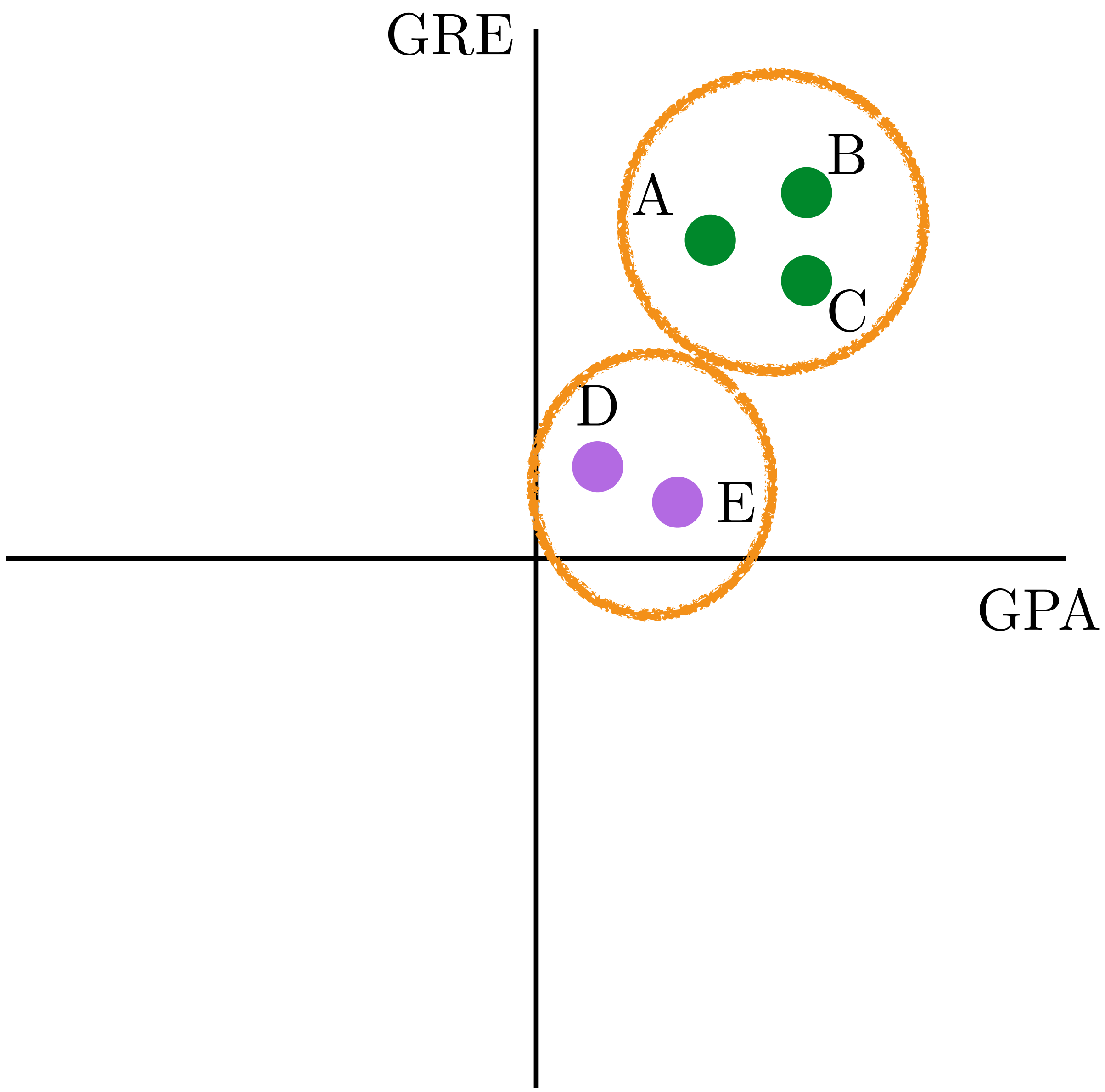
Distance Matrix



$$\mathbf{D} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 3 & 4 \\ 1 & 0 & 1 & 5 & 5 \\ 1 & 1 & 0 & 4 & 3 \\ 3 & 5 & 4 & 0 & 1 \\ 4 & 5 & 3 & 1 & 0 \end{pmatrix} \end{array}$$

Distance matrices are ***symmetric***

Distance Matrix



$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 1 & 3 & 4 \\ 1 & 0 & 1 & 5 & 5 \\ 1 & 1 & 0 & 4 & 3 \\ 3 & 5 & 4 & 0 & 1 \\ 4 & 5 & 3 & 1 & 0 \end{pmatrix}$$

	A	B	C	D	E
A	0	1	1	3	4
B	1	0	1	5	5
C	1	1	0	4	3
D	3	5	4	0	1
E	4	5	3	1	0

Distance can help us find groups of similar objects (***clusters***)

Other Norms

- 1-Norm (Manhattan/CityBlock/Taxicab distance)

$$\| \mathbf{x} \|_1 = | x_1 | + | x_2 | + \ldots + | x_n |$$

- ∞-Norm (Max Distance)

$$\| \mathbf{x} \|_\infty = \max\{| x_1 |, | x_2 |, \ldots, | x_n |\}$$

Norms in Statistics

- Standard deviation:

$$\frac{1}{\sqrt{n-1}}\|\mathbf{x}\|$$

← vector of centered data

$$s_x = \frac{1}{\sqrt{n-1}}\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

- Correlation Coefficient:

$$\frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\ \|\mathbf{y}\|}$$

← vectors of centered data

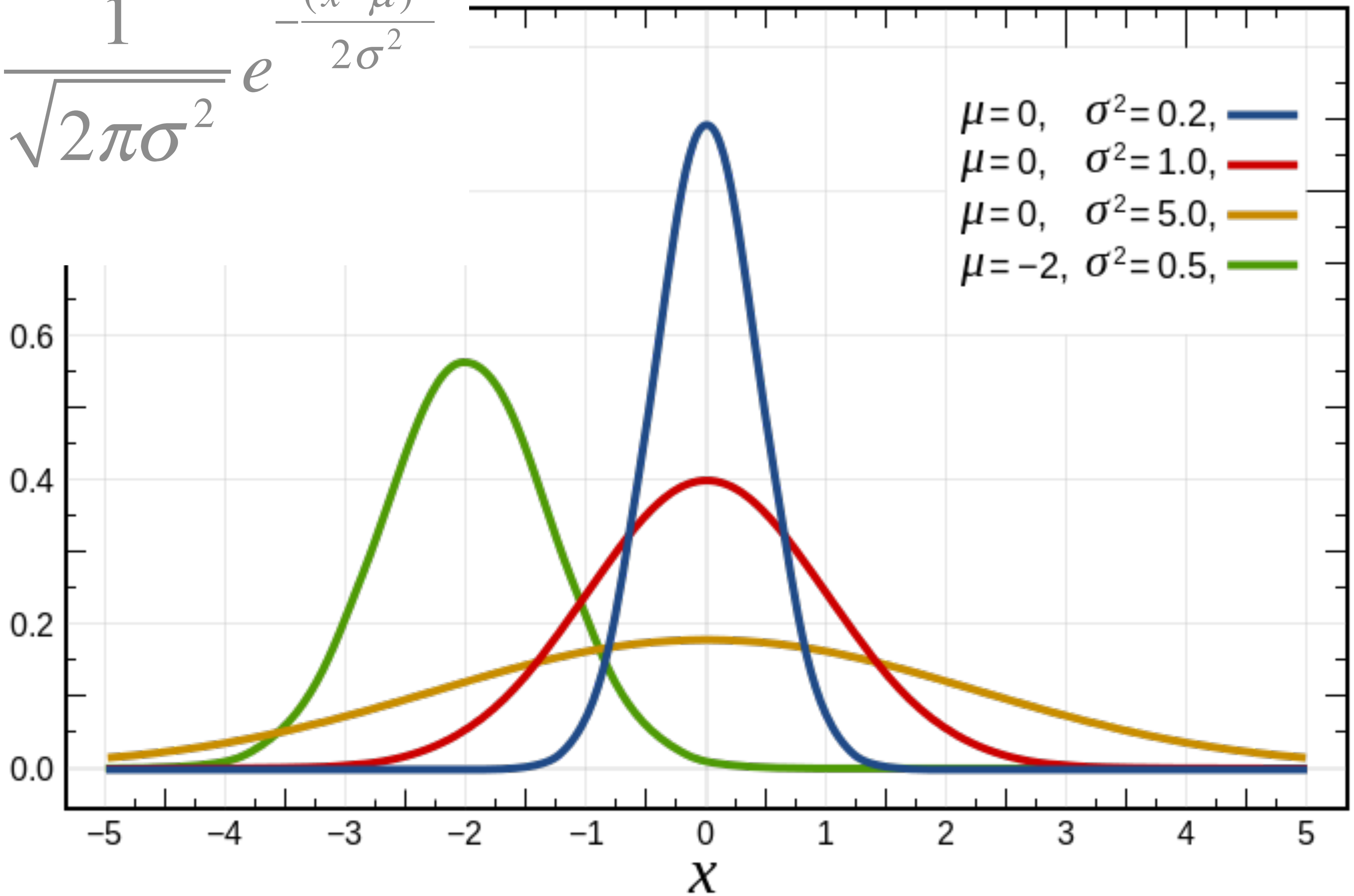
$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i-\bar{y})^2}}$$

Covariance and Correlation

The Multivariate Normal Distribution

Normal (Gaussian) Density Function

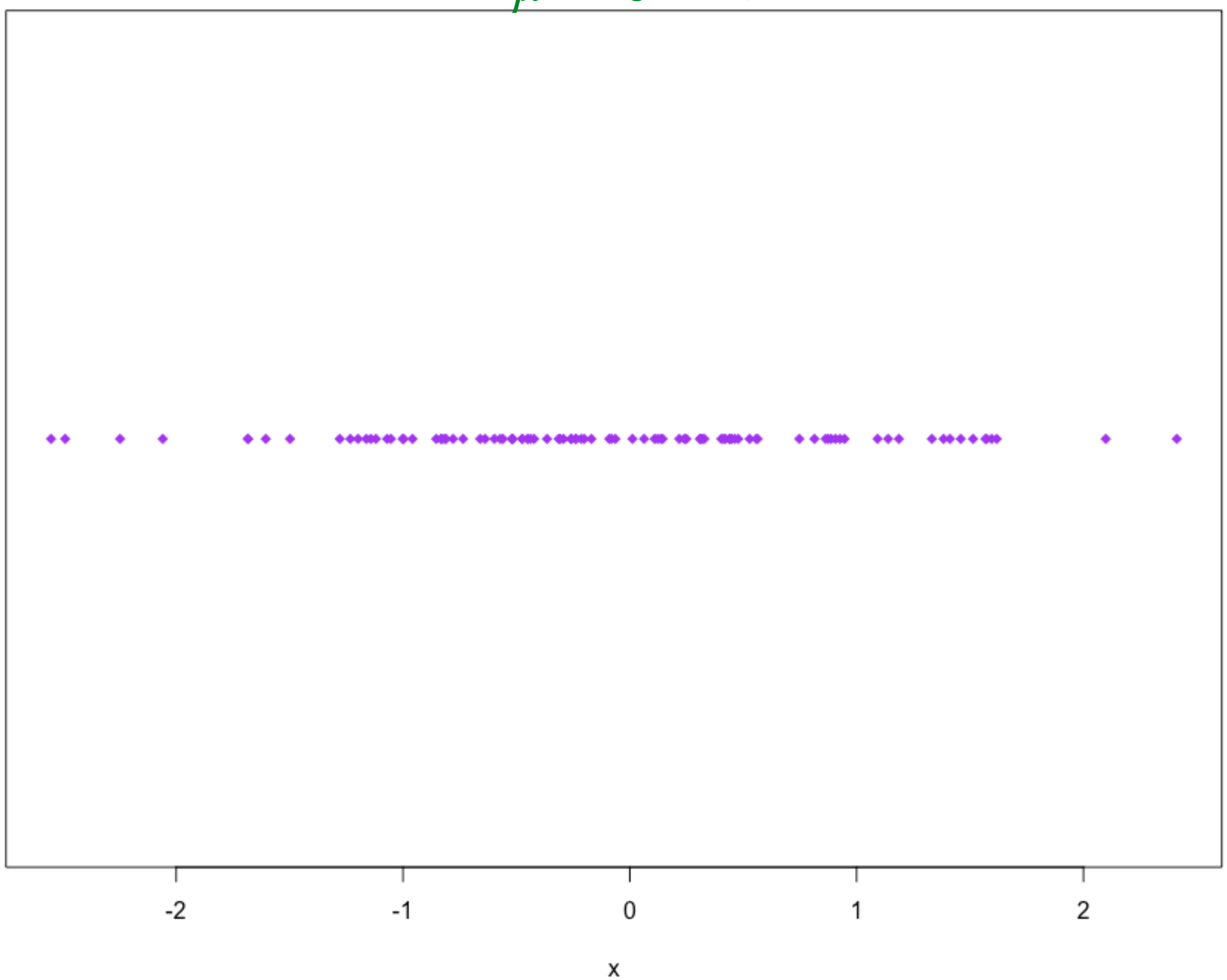
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Normal (Gaussian) Data Points

$x \sim N(0, 1)$

$\mu = 0$ $\sigma^2 = 1$



Normal (Gaussian) Data Points

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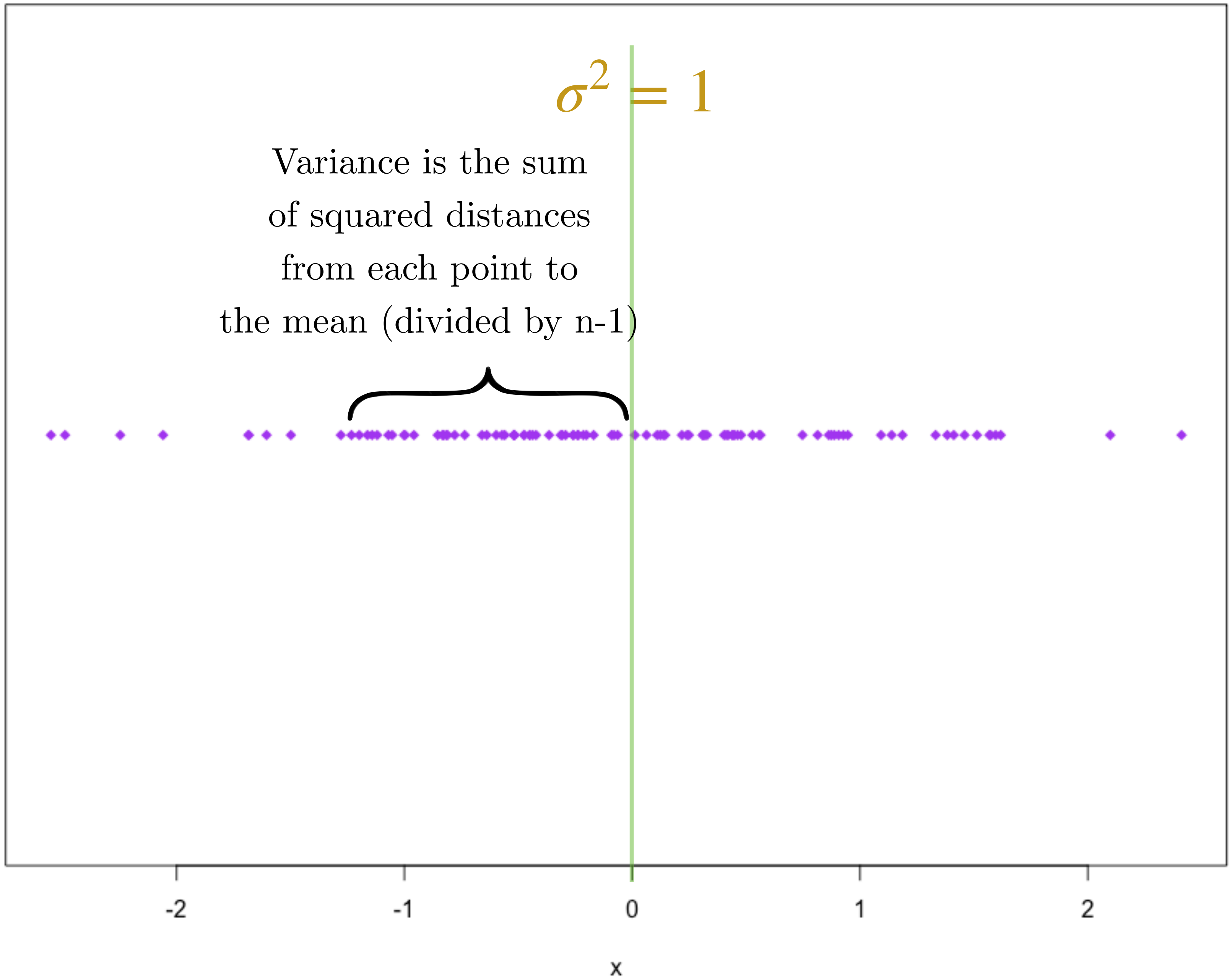


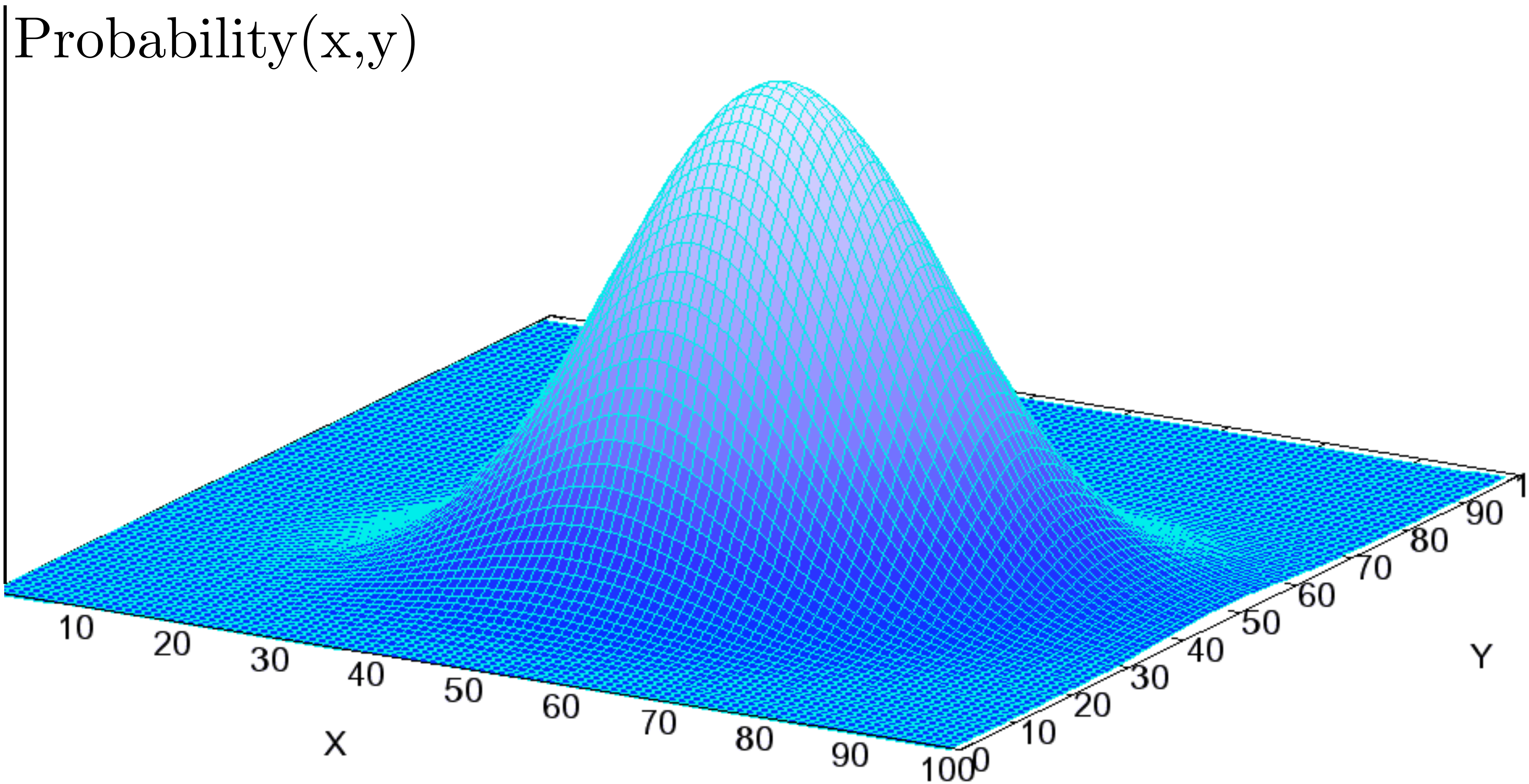
Covariance

- Covariance is a number that describes how two variables change together.

- If x increases/decreases, does y tend to increase/decrease? Covariance can be negative.

- Is a parameter of the **joint distribution** of x and y
  - joint distribution: how likely are we to see the pair (x,y) together?

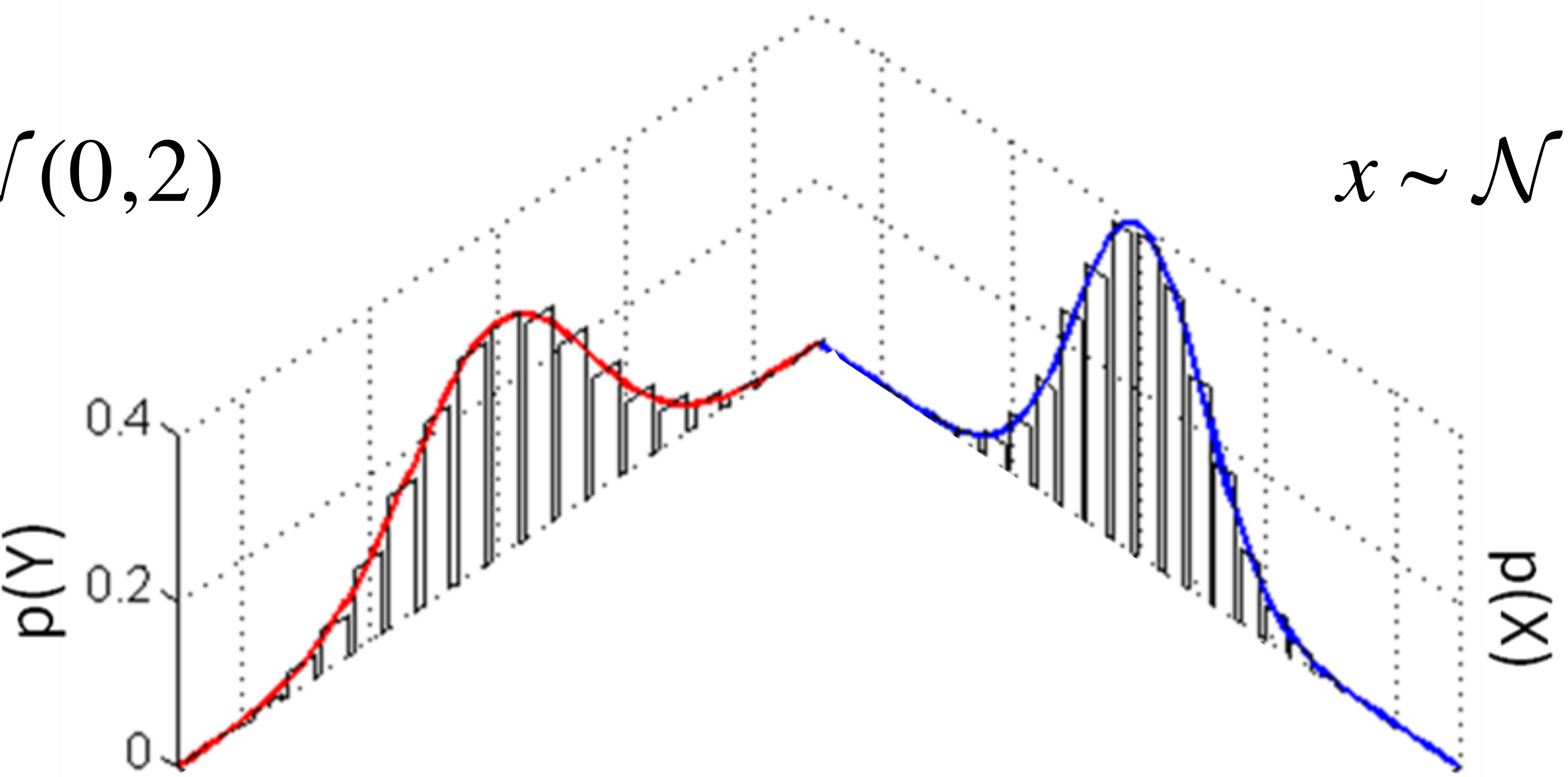
Joint Distribution of (x,y)



Multivariate Normal Distribution

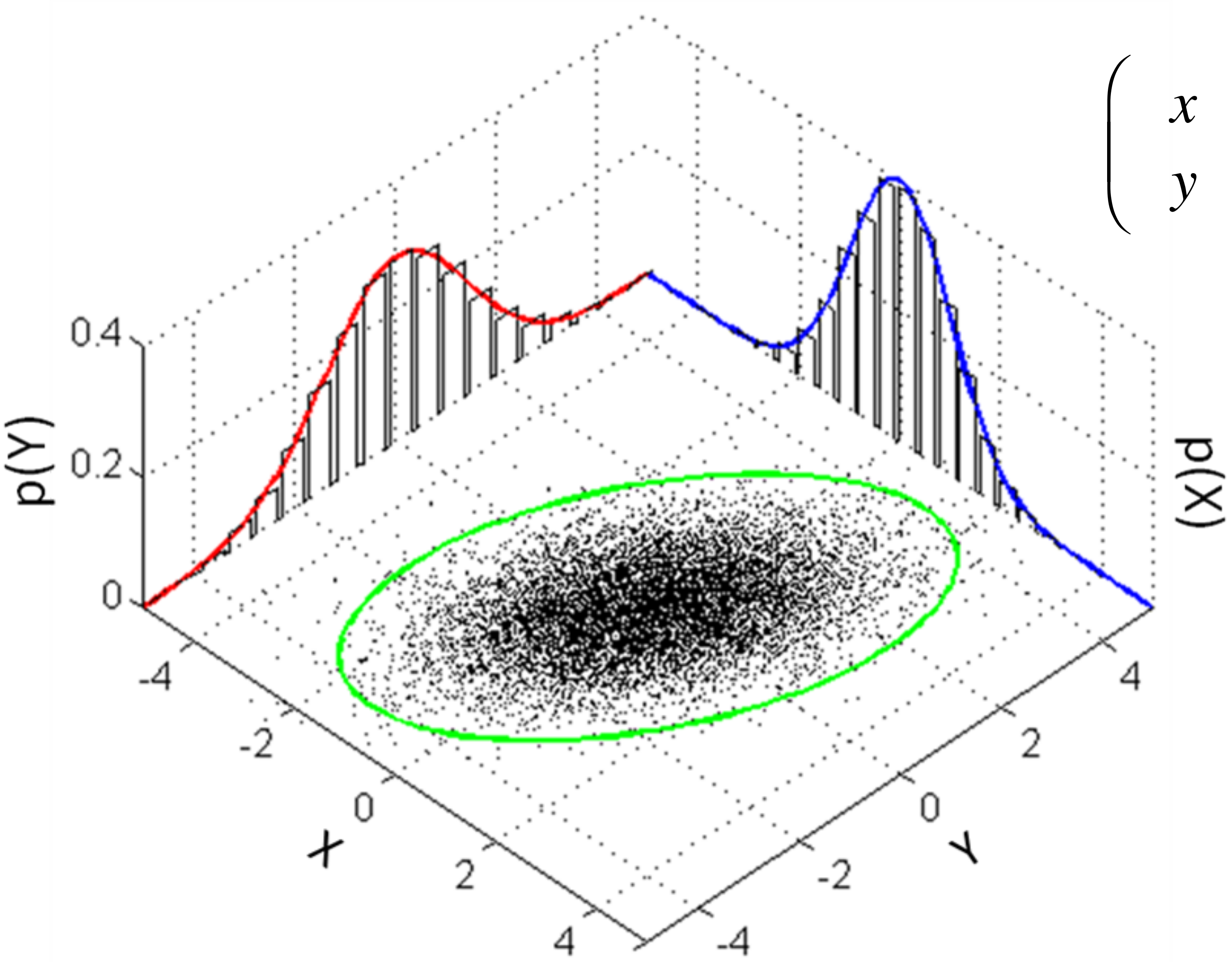
Suppose x and y are normally distributed

$y \sim \mathcal{N}(0,2)$ $x \sim \mathcal{N}(0,1)$



Multivariate Normal Distribution

The vector (x,y) is multivariate normally distributed



$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\substack{\boldsymbol{\mu} \\ \text{mean} \\ \text{vector}}}, \underbrace{\begin{pmatrix} 1 & 3/5 \\ 3/5 & 2 \end{pmatrix}}_{\substack{\boldsymbol{\Sigma}_{p \times p} \\ \text{covariance} \\ \text{matrix}}} \right)$$

Multivariate Normal Distribution

The vector (x,y) is multivariate normally distributed

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\mu}, \underbrace{\begin{pmatrix} 1 & 3/5 \\ 3/5 & 2 \end{pmatrix}}_{\Sigma_{p \times p}} \right)$$

μ mean vector

$\Sigma_{p \times p}$ covariance matrix

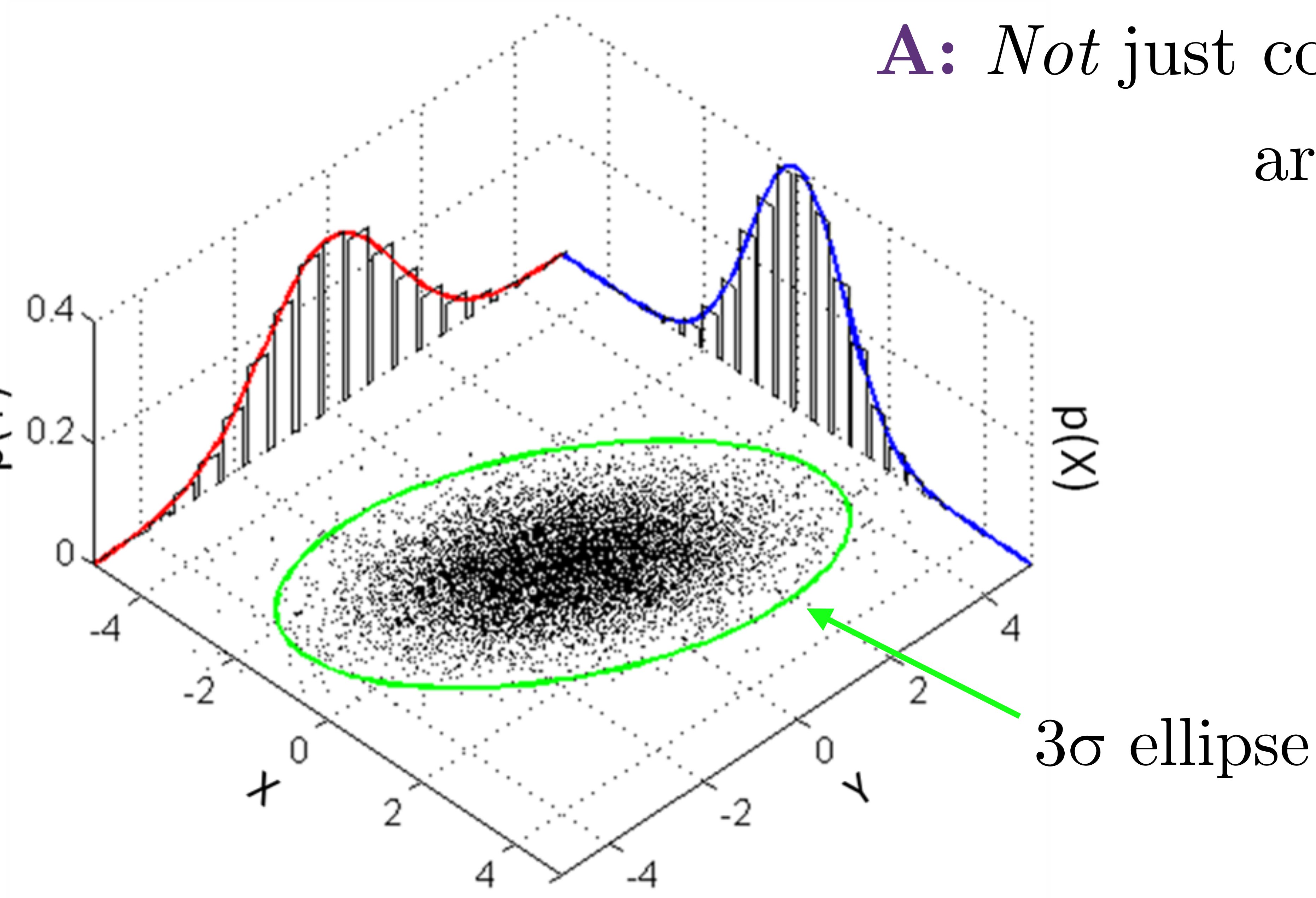
Covariance Matrix Fun Facts

- Variances of each variable on the main diagonal
- Covariances of each pair of variables on the off diagonal
- Always symmetric

Multivariate Normal Distribution

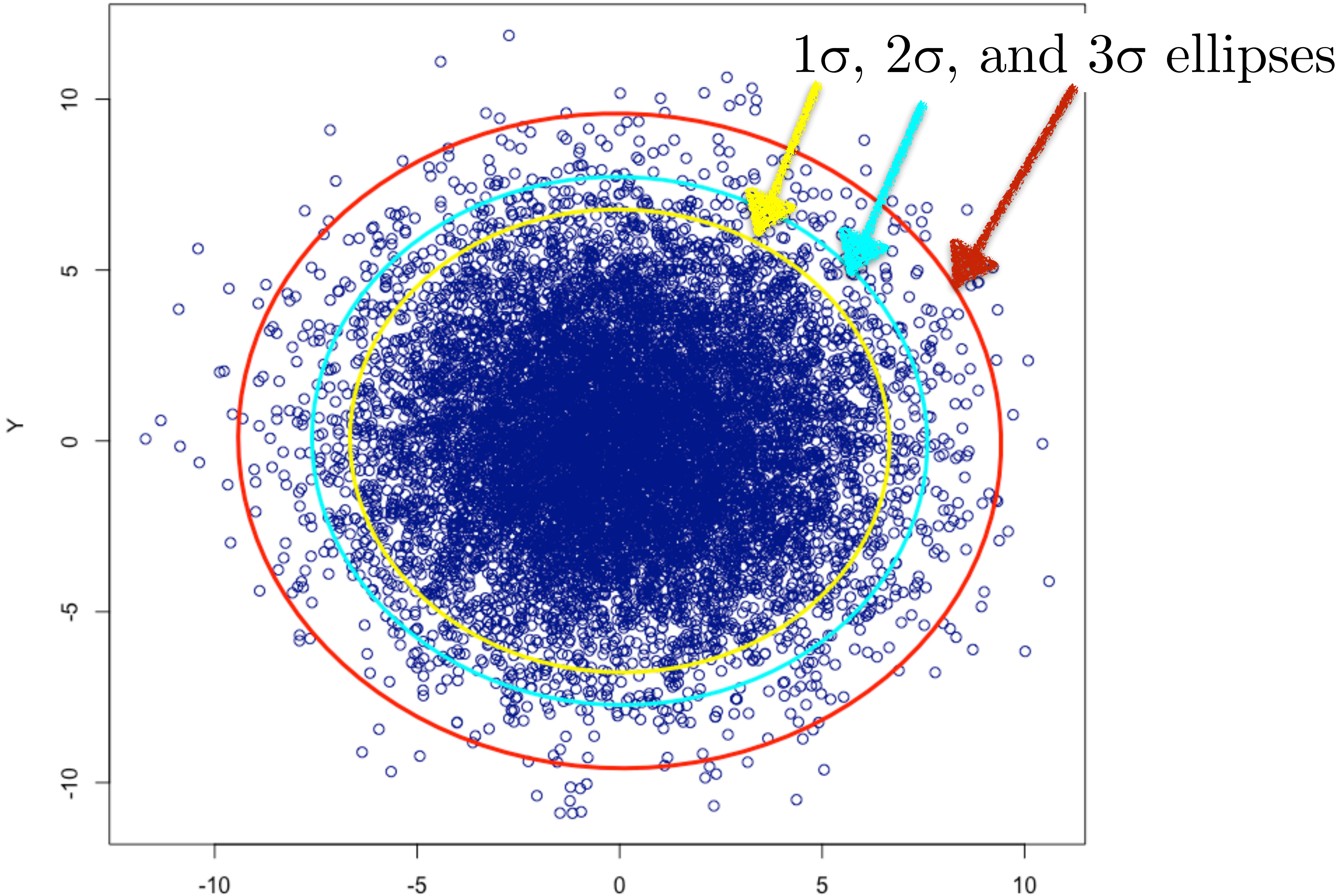
Q: How can we characterize a point as ***rare***?

A: *Not* just constant distance around the mean!



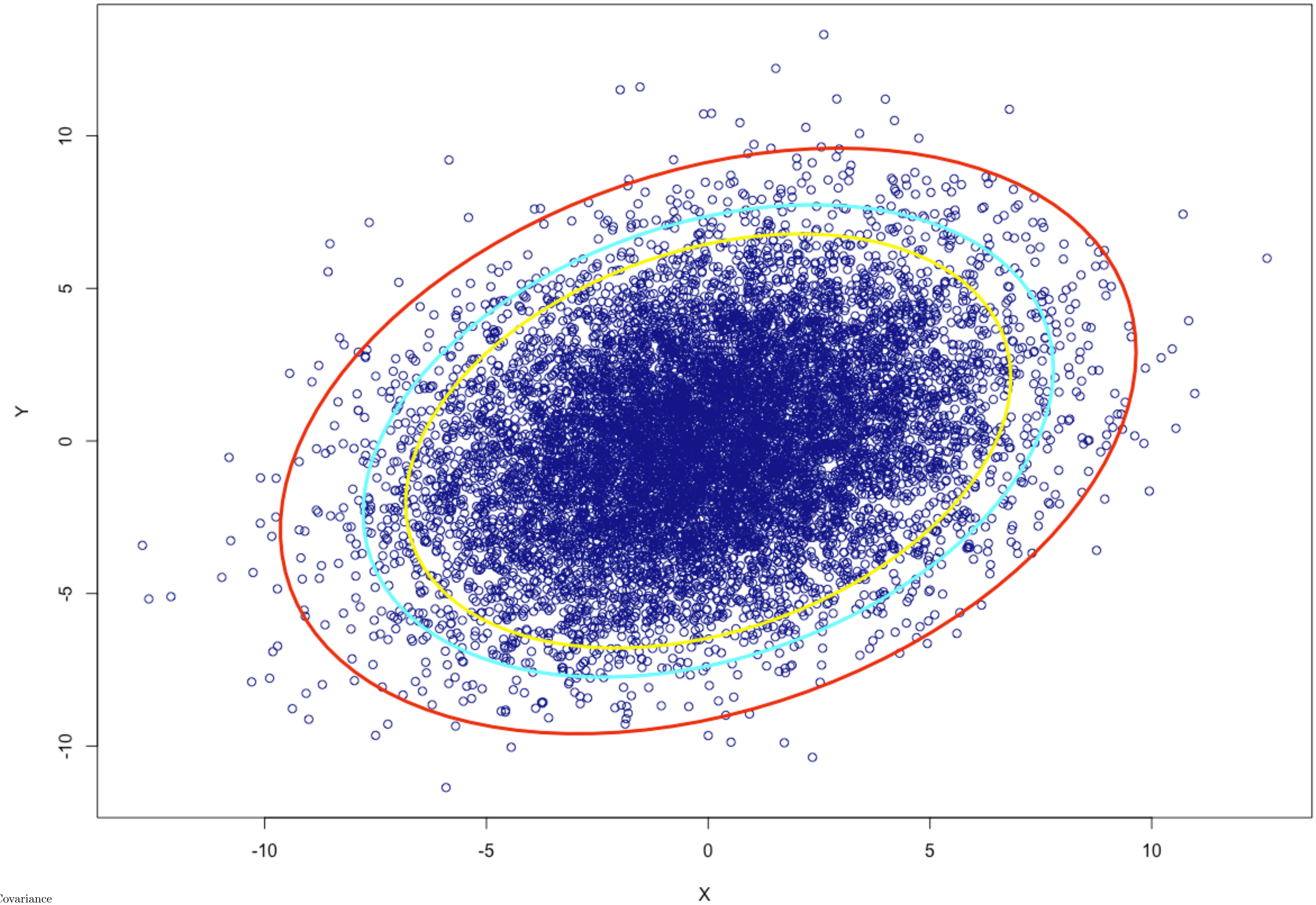
Multivariate Normal Distribution

Var(X)=Var(y) and Covariance = 0



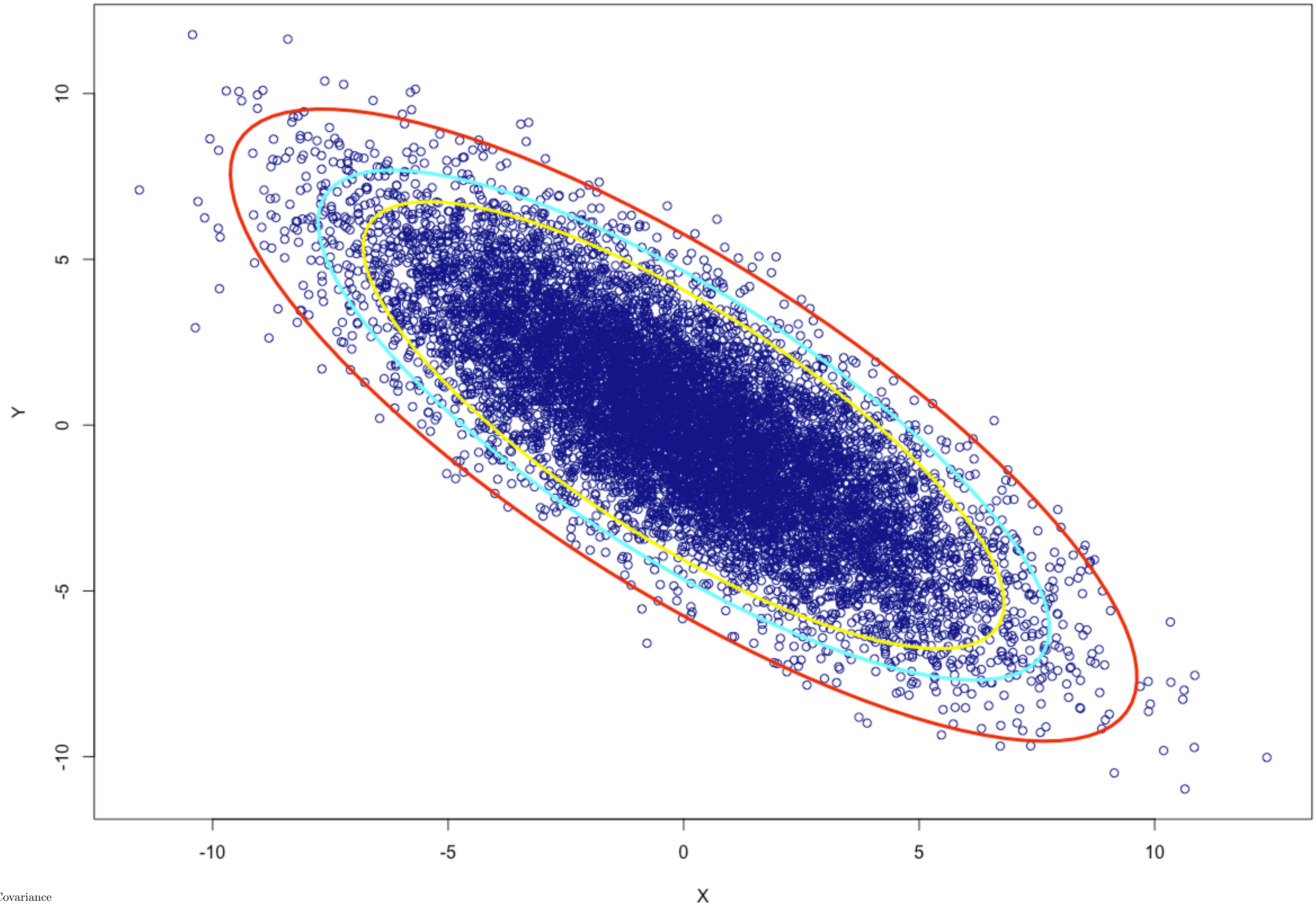
Multivariate Normal Distribution

Var(X)=Var(y) and Covariance = 4



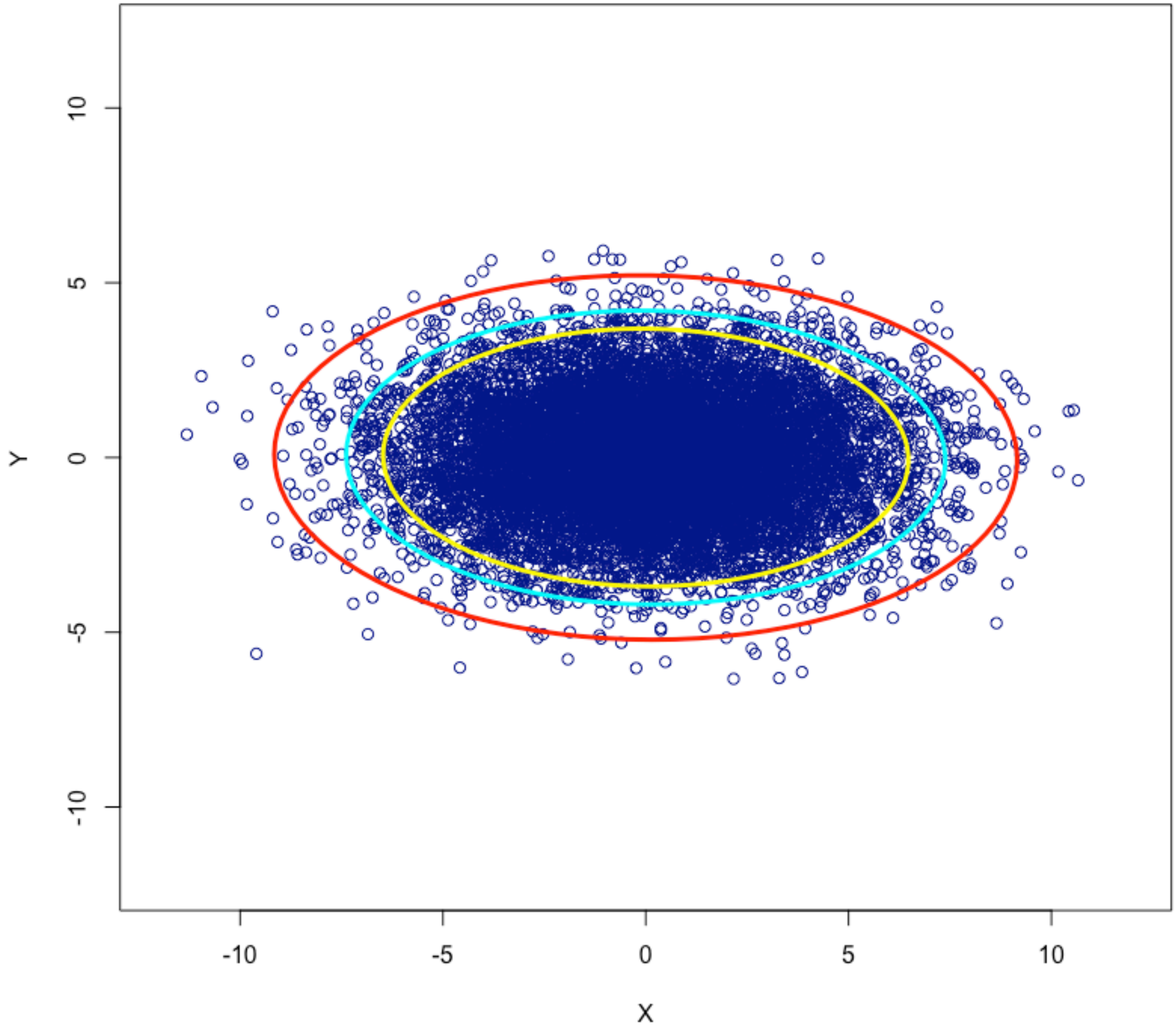
Multivariate Normal Distribution

Var(X)=Var(y) and Covariance = -8

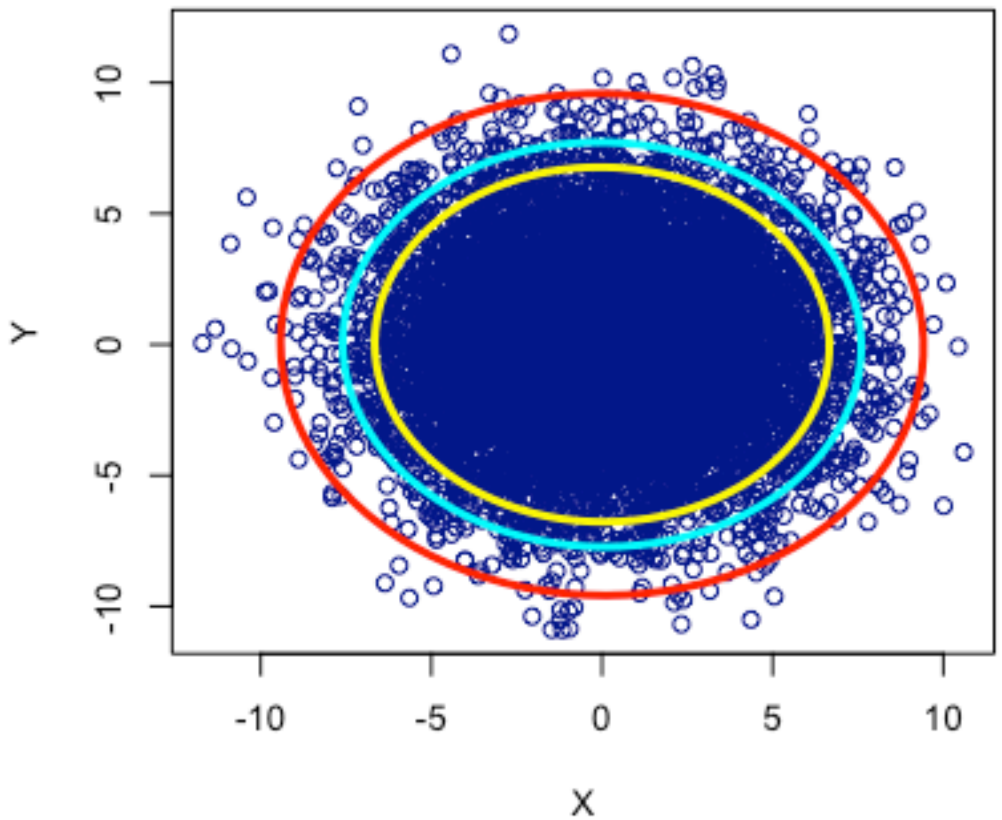


Multivariate Normal Distribution

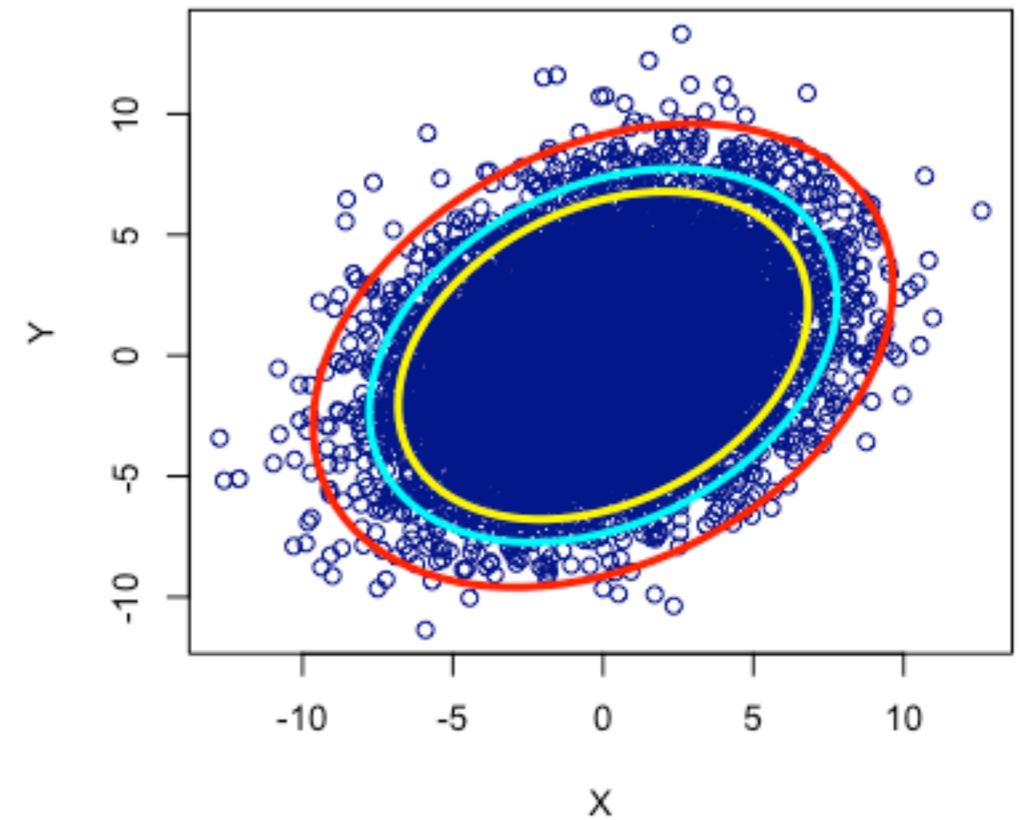
Var(X)=3*Var(y) and Covariance = 0



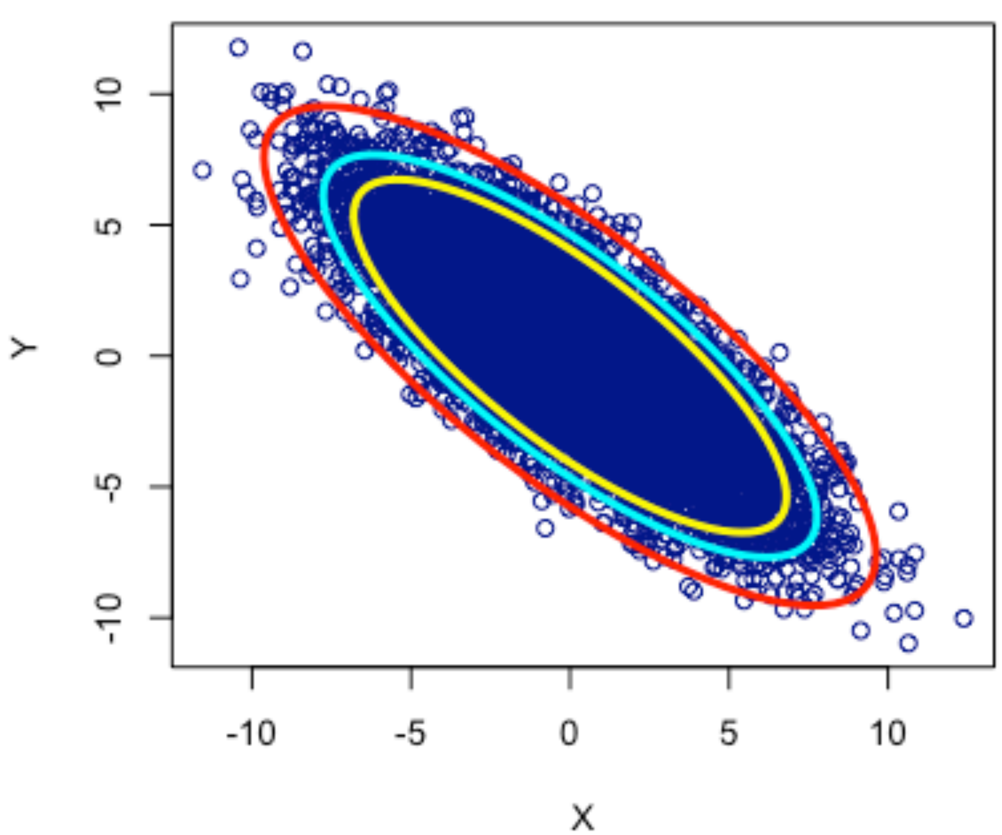
Var(X)=Var(y) and Covariance = 0



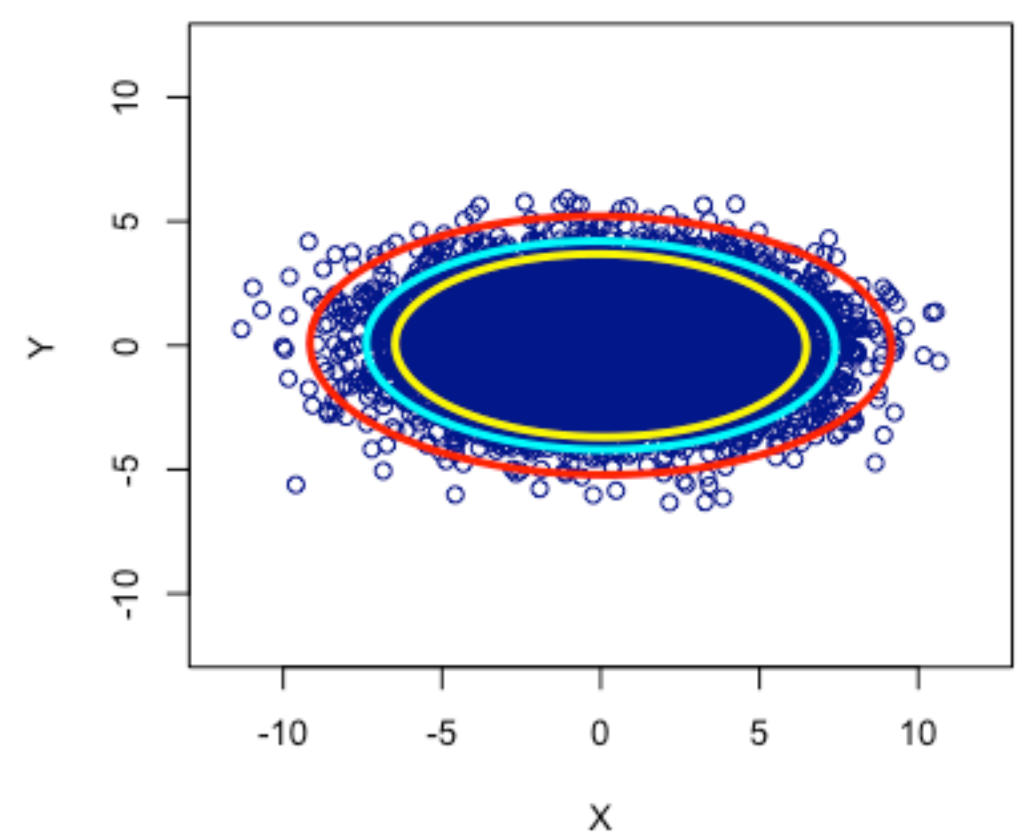
Var(X)=Var(y) and Covariance = 4



Var(X)=Var(y) and Covariance = -8



Var(X)=3*Var(y) and Covariance = 0



Covariance

Covariance is calculated from the data:

vectors of
<u>centered</u> data

$$Cov(x,y) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = \frac{1}{n-1}\mathbf{x}^T\mathbf{y}$$

When covariance is positive:

x larger than mean, y tends to be larger than the mean
x smaller than the mean, y tends to be smaller than the mean

When covariance is negative:

x larger than mean, y tends to be smaller than the mean
x smaller than the mean, y tends to be larger than the mean

The units will have a strong effect on this number so we ***cannot interpret magnitude, only sign***!!

Correlation

Correlation is the covariance of the standardized data:

$$Corr(x,y) = \frac{1}{n-1}\sum_{i=1}^{n}\frac{(x_i-\bar{x})}{s_x}\frac{(y_i-\bar{y})}{s_y} = \frac{1}{n-1}\mathbf{x}^T\mathbf{y}$$

vectors of <u>standardized</u> data

As we already know, correlation is between -1 and 1 and its magnitude measures the tightness of a relationship.

NOT THE SLOPE

Primer Tutorials
(Prioritized)

http://www4.ncsu.edu/~slrace/LAprimer/index.html

- Tutorial 2 (Basic terminology) **12 minutes**
- Tutorials 3-4 (Matrix Arithmetic) **33 minutes**
- Tutorial 5 (Applications of Arithmethic) **17 minutes**
- Tutorial 13 (Basic Matrix Algebra) **12 minutes**
- Tutorial 15 (Norms&Distance Measures) **27 minutes**

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101 minutes