Orthogonality - Worksheet

1. What is the cosine of the angle between $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

$$\frac{x^T y}{\|x\| \, \|y\|} = \frac{1}{\sqrt{2}}$$

2. Are vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ orthogonal? How do you know?

yes b/c $v_1^T v_2 = 0$

3. What are the two conditions necessary for a collection of vectors to be orthonormal?

1) mutually orthogonal ($x_i^T x_j = 0$ for $i \neq j$)

2) each is a unit vector (length 1) ($x_i^T x_i = 1 \Leftrightarrow \|x_i\| = 1$ for all i)

4. Briefly explain why an orthonormal basis is important.

Anything else would distort the data and would not preserve distances or angles between vectors. We also want to focus on the coordinates and perhaps see which coordinates are "bigger." This only makes sense if the coordinates reflect distance from the origin.

1. Let $\mathbf{U} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 0 & -2 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ -2 & 1 & 0 & 2 \end{pmatrix}$

a. Show that $\mathbf{U}$ is an orthogonal matrix.

Compute $U^T U = I$

since U is square and $U^T U = I$, U is orthogonal

b. Let $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Solve the equation $\mathbf{Ux} = \mathbf{b}$.

$$Ux = b$$
$$\underbrace{U^T U}_{I} x = U^T b$$
$$I x = U^T b$$
$$\boxed{x = U^T b}$$

since b is column of all 1's, just sum the rows of U^T

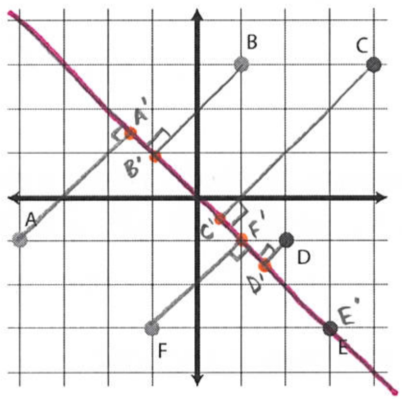
$$x = \frac{1}{3}\begin{pmatrix} -1 \\ 5 \\ 3 \\ 1 \end{pmatrix}$$

2. Find two vectors which are orthogonal to $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

many answers possible (infinitely many)

$a = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

3. Draw the orthogonal projection of the points onto the subspace $span \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$



List of Key Words.

cosine
orthogonal
orthonormal
orthogonal Matrix
orthonormal basis
orthogonal projection