Eigenvectors and Intro to PCA - Worksheet

Part One

1. Show that **v** is an eigenvector of **A** and find the corresponding eigenvalue:

   a. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad \mathbf{v} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ $\quad Av = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ $\quad \lambda = -1$

   b. $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 6 & 0 \end{pmatrix}$ $\quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\quad Av = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$ $\quad \lambda = -3$

   c. $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & -7 \end{pmatrix}$ $\quad \mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\quad Av = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$ $\quad \lambda = 3$

2. Can a rectangular matrix have eigenvalues/eigenvectors?

   no. The eigenvalue equation wouldn't even make sense

   $$\underset{(m\times n)\,(n\times 1)}{A\,x} = \underset{\to (m\times 1)}{\lambda\, x}$$ } vector x can't be n×1 on right and m×1 on the left! m=n ⇒ SQUARE

3. For the following matrix, determine the eigenvalue associated with the given eigenvector.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \lambda = 0$$

   From this eigenvalue, what can you conclude about the matrix **A**?

   The matrix is not full rank ≡ columns linearly dependent

4. The matrix **M** has eigenvectors **u** and **v**. What is λ_1, the first eigenvalue for the matrix **M**?

$$\mathbf{M} = \begin{pmatrix} -1 & 1 \\ 6 & 0 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

   $\lambda_1 = -3$

5. For the previous problem, is the specific eigenvector (**u** or **v**) the *only* eigenvector associated with λ_1?

   no, any scalar multiple of an eigenvector is also an eigenvector associated w/ same eigenvalue.

Part Two

1. For the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 4 \\ -1 & 5 \end{pmatrix},$$

a. Verify that the pairs

$$(\lambda_1 = 4, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \quad \text{and} \quad (\lambda_2 = 1, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix})$$

are eigenpairs of **A**.

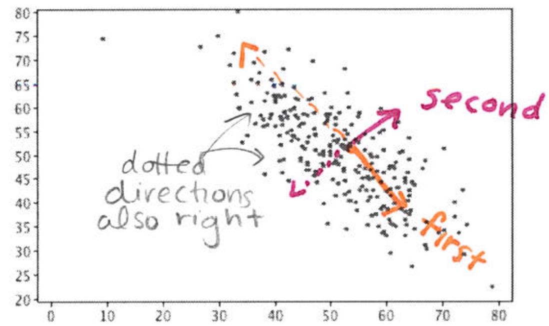
$Av_1 = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v_1$ $\quad$ $Av_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 1v_2$

b. For each pair in part a, provide *another* eigenvector associated with that eigenvalue.

$8v_1 = \begin{pmatrix} 8 \\ 8 \end{pmatrix}$ $\quad$ $3v_2 = \begin{pmatrix} 12 \\ 3 \end{pmatrix}$

Part Three

1. For the following data plot, take your best guess and draw the direction vectors of the first and second principal components (the eigenvectors of the covariance matrix).



2. Suppose your data contained the variables *VO2_max*, *mile pace*, and *weight* in that order. The first principal component for this data is the eigenvector of the covariance matrix.

$$\begin{pmatrix} 0.69 \\ 0.61 \\ -0.38 \end{pmatrix}.$$

What would be the sign of the coordinate along this basis vector for an individual that had above average *VO2_max*, above average *mile pace* and below average *weight*? Explain. positive.

Coordinate for this dimension=
$0.69\,(VO_2 max) + 0.61\,(\text{mile Pace}) - 0.38\,(\text{weight})$
using centered data. so, we have
$0.69(+) + 0.61(+) - 0.38(-)$
which must be positive

List of Key Words/Phrases.

- eigenvalue
- eigenvector
- eigenpair
- eigenspace
- $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$
- diagonalization
- eigenvalues of symmetric matrices
- principal components
- directional variance
- proportion of variance
- correlation matrix
- covariance matrix
- orthogonal projection
- PCA loadings
- biplot
- zero eigenvalues