Worksheet - Lecture 14

Partitioned Matrix Arithmetic

1. Suppose I want to compute the matrix product $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ where $\mathbf{U}$ is $n \times r$, $\mathbf{D}$ is an $r \times r$ diagonal matrix, $\mathbf{D} = diag\{\sigma_1, \sigma_2, \dots, \sigma_r\}$, and $\mathbf{V}^T$ is $r \times p$. *(Side note: we will quite often want to compute such a matrix product – this is the form of the singular value decomposition (SVD)! The following exercise is not just for fun - what you end up with in part b is exactly how we will want to write the SVD to best understand how it works.)*

   a. Using what you know about multiplication by diagonal matrices, if we view the matrix $\mathbf{U}$ as a collection of columns,
   $$\mathbf{U} = (\mathbf{U}_1|\mathbf{U}_2|\mathbf{U}_3|\dots|\mathbf{U}_r)$$
   then how would I write the same partition of the matrix $\mathbf{UD}$?
   $$\mathbf{UD} = (?|?|?|\dots|?)$$

   Keep in mind that when multiplying matrices/vectors by scalars, it is always preferable to write the scalar first ($\sigma\mathbf{x}$ rather than $\mathbf{x}\sigma$)

   b. Now, using the above representation for $\mathbf{UD}$, what happens when I multiply by the matrix $\mathbf{V}^T$, viewed as a collection of rows,
   $$\mathbf{V}^T = \begin{pmatrix} \mathbf{V}_1^T \\ \hline \mathbf{V}_2^T \\ \hline \mathbf{V}_3^T \\ \hline \vdots \\ \hline \mathbf{V}_r^T \end{pmatrix} ?$$
   $$\mathbf{UDV}^T = ?$$

   (Hint: your answer should be a sum. Each term in the sum should be an outer product.)

2. Consider

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & 4 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ 2 & -1 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 & 3 \\ -1 & -2 & 0 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

Partition these into submatrices (regions/blocks) conformably for multiplication as follows:

$$\mathbf{A} = \left(\begin{array}{c|c|c} \mathbf{A}_{00} & \mathbf{a}_{01} & \mathbf{A}_{02} \\ \hline \mathbf{a}_{10}^T & \alpha_{11} & \mathbf{a}_{12}^T \\ \hline \mathbf{A}_{20} & \mathbf{a}_{21} & \mathbf{A}_{22} \end{array}\right) \qquad \mathbf{x} = \left(\begin{array}{c} x_0 \\ \hline \chi_1 \\ \hline x_2 \end{array}\right)$$

Where $\mathbf{A}_{00}$ is a 3×3 matrix, $\mathbf{x}_0 \in \mathbb{R}^3$, α_{11} is a scalar and χ_1 is a scalar. Show with lines how $\mathbf{A}$ and $\mathbf{x}$ are partitioned below:

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & 4 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ 2 & -1 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 & 3 \\ -1 & -2 & 0 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

3. For all matrices $\mathbf{A}_{n \times k}$ and $\mathbf{B}_{k \times n}$, show that the block matrix

$$\mathbf{L} = \begin{pmatrix} \mathbf{I} - \mathbf{BA} & \mathbf{B} \\ 2\mathbf{A} - \mathbf{ABA} & \mathbf{AB} - \mathbf{I} \end{pmatrix}$$

satisfies the property $\mathbf{L}^2 = \mathbf{I}$. *Hint: Perform block matrix multiplication for each of the four separate blocks in the result, simplifying each expression as much as possible.*