Review Packet 1

1. For each of the following, write the vector or matrix that is specified:

   a. $\mathbf{e}_3 \in \mathbb{R}^4$

   b. $\mathbf{D} = diag\{2, \sqrt{3}, -1\}$

   c. $\mathbf{e} \in \mathbb{R}^3$

   d. $\mathbf{I}_2$

2. For each of the following matrices and vectors, give their dimension. Label each as a matrix or vector. For each matrix, indicate whether the matrix is square or rectangular.

   a.
$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

   b.
$$\mathbf{h} = \begin{pmatrix} -1 \\ -4 \\ 1 \\ 2 \end{pmatrix}$$

   c.
$$\mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \\ B_{41} & B_{42} & B_{43} \end{pmatrix}$$

   d.
$$\mathbf{A} = [A_{ij}] \quad \text{where } i = 1,2,3 \quad \text{and } j = 1,2$$

3. Specify whether the following augmented matrices are in row-echelon form (REF), reduced row-echelon form (RREF), or neither:

   a. $\left(\begin{array}{ccc|c} 3 & 2 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right)$ ________________

   b. $\left(\begin{array}{ccc|c} 3 & 2 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 4 & 0 & 0 \end{array}\right)$ ________________

   c. $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ ________________

   d. $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ ________________

4. *(True/False)* The normal equations are used to find the ordinary least-squares solution to an inconsistent system of equations.

5. If the matrix equation $\mathbf{Mv} = \mathbf{b}$ is inconsistent, what alternative equation should I solve to find a solution $\widehat{\mathbf{v}}$ such that $\mathbf{M}\widehat{\mathbf{v}} = \widehat{\mathbf{b}}$ is as close to $\mathbf{b}$ as possible in the sense that it minimizes the sum of squared error:

$$SSE = \sum_{i=1}^{n} (\widehat{\mathbf{b}}_i - \mathbf{b}_i)^2$$

6. Answer the following questions about each matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 5 & 2 \\ 5 & 3 & 1 \\ 2 & 1 & -2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

a. Is the matrix square?

A__________ **B**__________

b. What is the transpose of the matrix?

$\mathbf{A}^T =$ $\mathbf{B}^T =$

c. Is the matrix symmetric?

A__________ **B**__________

d. If possible, name the diagonal elements of the matrix.

A__________ **B**__________

e. If possible, compute the Trace of the matrix.

A__________ **B**__________

f. Can the product **AB** be computed? If so, what is the size of the result?

____

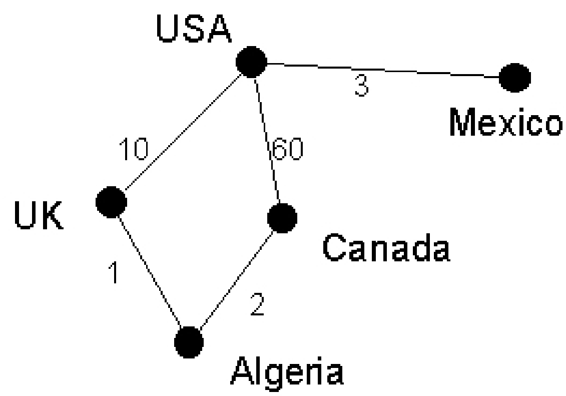
g. Can the product **BA** be computed? If so, what is the size of the result?

____

h. Can the product $(\mathbf{B}_{\star 3})^T (\mathbf{A}_{3\star})^T$ be computed? If so, what is the result?

7. What is the inverse of the matrix $\mathbf{D} = \sigma \mathbf{I}_3$?

8. For the following graph, number the nodes and write the corresponding adjacency matrix:



9. Compute the outer product $\mathbf{x}\mathbf{y}^T$ where

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathbf{x}\mathbf{y}^T =$$

10. Can you say anything about the rank of an outer product in general? Explain your answer.

11. Briefly explain what it means for a matrix to be full rank.

12. For the following augmented matrices, circle the pivot elements and give the rank of the coefficient matrix along with the number of free variables.

a. $\left(\begin{array}{cccc|c} 3 & 2 & 1 & 1 & 2 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 5 \end{array}\right)$ rank=_______ # free var=_______

b. $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ rank=_______ # free var=_______

c. $\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ rank=_______ # free var=_______

13. Write the vector $\mathbf{v}$ as a linear combination of each given $\mathbf{x}$ and $\mathbf{y}$, if possible.

$$\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

a. $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\mathbf{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b. $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\mathbf{y} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

c. $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\mathbf{y} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

14. Suppose we measure the heights of 10 people, $person_1, person_2, \ldots, person_{10}$.

a. If we create a matrix $\mathbf{S}$ where

$$S_{ij} = height(person_i) - height(person_j)$$

is the matrix $\mathbf{S}$ symmetric? What is the trace($\mathbf{S}$)?

b. If instead we create a matrix $\mathbf{G}$ where

$$G_{ij} = [height(person_i) - height(person_j)]^2$$

is the matrix $\mathbf{G}$ symmetric? What is the trace($\mathbf{G}$)?

15. For the matrix $\mathbf{H} = \mathbf{X}(\mathbf{X}^T\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}^T\mathbf{\Sigma}^{-1}$, use the properties of matrix arithmetic to show that

a. $\mathbf{H}^2 = \mathbf{H}$

b. $\mathbf{H}(\mathbf{I} - \mathbf{H}) = \mathbf{0}$

16. Let

$$\mathbf{U} = (\mathbf{U}_1|\mathbf{U}_2|\mathbf{U}_3|\dots|\mathbf{U}_p) \quad \text{and} \quad \mathbf{V}^T = \begin{pmatrix} \mathbf{V}_1^T \\ \hline \mathbf{V}_2^T \\ \hline \mathbf{V}_3^T \\ \hline \vdots \\ \hline \mathbf{V}_p^T \end{pmatrix}$$

Write the matrix product $\mathbf{U}\mathbf{V}^T$ in terms of the columns of $\mathbf{U}$ and the rows of $\mathbf{V}^T$.

17. Suppose that $\mathbf{u}$ is a unit vector. Then, $\|\mathbf{u}\|_2 =?$

18. Let $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$. Compute the following:

a. $\|\mathbf{x}\|_2$

b. $\|\mathbf{y}\|_1$

c. $\|\mathbf{y}\|_\infty$

d. $\|\mathbf{x} - \mathbf{y}\|_2$

e. $\|\mathbf{x} - \mathbf{y}\|_1$

19. Suppose we have a dataset containing survey data. Individuals were asked to respond 'yes'=1 or 'no'=0 to twenty potential political referendums. Let $\mathbf{a}$ be the vector containing the numerical responses of Individual A and let $\mathbf{b}$ be the vector containing the numerical responses of Individual B (so $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{20}$). Explain in words the interpretation of the quantity

$$\|\mathbf{a} - \mathbf{b}\|_1.$$

20. **Statistical Formulas Using Linear Algebra Notation.** Almost every statistical formula can be written in a more compact fashion using linear algebra. Most of the elementary formulas involve vector inner products or the Euclidean norm. To begin, we'll introduce the concept of *centering* the data. **Centering** the data means that the mean of a variable is subtracted from each observation. For example, if we have some variable, $\mathbf{x}$, and 3 observations on that variable:

$$\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

then obviously, $\bar{x} = 3$. The **centered** version of $\mathbf{x}$ would then be

$$\mathbf{x} - \bar{x}\mathbf{e} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

We simply subtract the mean from every observation so that the new mean of the variable is 0.

Most multivariate textbooks start by saying *"all variable vectors in this textbook are assumed to be centered to have mean zero unless otherwise specified"*. Looking at the most common statistical formulas helps us see why. Try to re-write the following formulas using linear algebra notation, using the vectors $\mathbf{x}$ and $\mathbf{y}$ to represent centered data:

$$\mathbf{x} = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ x_3 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ y_3 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix}$$

For this exercise, keep in mind the following linear algebra constructs, which you should be very familiar with by now:

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}$$

$$\mathbf{a}^T\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n$$

a. Sample standard deviation:

$$s = \frac{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}{\sqrt{n-1}} = \boxed{}$$

b. Sample covariance:

$$covariance(\mathbf{x}, \mathbf{y}) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \boxed{}$$

c. Correlation coefficient:

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}} = \boxed{}$$

21. **List of Key Words.** You should be comfortable with the following vocabulary:

linear
matrix
vector
scalar
A_{ij}
$\mathbf{A}_{\star j}$
$\mathbf{A}_{i\star}$
dimensions
diagonal element
square matrix
rectangular matrix
network
graph
adjacency matrix
correlation matrix
transpose
symmetric matrix
trace
diagonal matrix
identity matrix
upper triangular matrix
lower triangular matrix
matrix addition
matrix subtraction
scalar multiplication
inner product
matrix product

linear combination
outer product
matrix inverse
systems of equations
row operations
row-echelon form
pivot element
Gaussian elimination
Gauss-Jordan elimination
reduced row-echelon form
rank
unique solution
infinitely many solutions
inconsistent
back-substitution
residual error
least squares
normal equations
least squares solution
parameter estimate
linearly independent
linearly dependent
full rank
perfect multicollinearity
severe multicollinearity
invertible
nonsingular

Distributive Property
$\mathbf{A}(\mathbf{B}+\mathbf{C}) =?$
$(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D}) =?$
Associative Property
Transpose of Product
$(\mathbf{ABC})^T =?$
$(\alpha\mathbf{A})^T =?$
Inverse of Transpose, $\mathbf{A}^{-T}$
Partitioned Matrices
Vector Norm
Magnitude/Length
2-norm
$\|\mathbf{x}\|_2$
$\sqrt{\mathbf{x}^T\mathbf{x}}$
Euclidean Norm
Euclidean Distance
Unit vector
Create unit vector
1-norm
$\|\mathbf{x}\|_1$
Manhattan distance
Taxicab distance
Cityblock distance
$\|\mathbf{x}\|_\infty$
Max Distance
Mahalanobis distance