

Chapter 3

Applications of Matrix Multiplication

Systems of Equations

Matrix multiplications with a set of unknown variables creates a **system of equations**. For example, if

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 2 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 10 \\ 15 \\ 6 \end{pmatrix}$$

Then the matrix equation $\mathbf{Ax} = \mathbf{b}$ creates the system

$$\begin{cases} x_1 + 2x_2 + 3x_3 + x_4 = 10 \\ + 3x_2 + 2x_3 + x_4 = 15 \\ x_1 + x_2 + x_3 + 4x_4 = 6 \end{cases}$$

- Systems of equations pop-up in all types of data applications from regression analysis to optimization.

Example - Trail Mix

A large manufacturing company has recently signed a deal to manufacture trail mix for a well-known food label. This label makes 3 versions of its product - one for airlines, one for grocery stores, and one for gas stations. Each version has a different mixture of peanuts, raisins, and chocolate which serves as the base of the trail mix. The base mixtures are made in 15 kg batches and sent to a second building for packaging.

Example - Trail Mix

The table contains the information about the mixes, each row containing the recipe for a 15 kg batch.

	Raisins (kg/batch)	Peanuts (kg/batch)	Chocolate (kg/batch)
Airline (a)	7	6	2
Grocery (g)	2	5	8
Gas Station (s)	6	4	5
Storage (kg)	380	500	620
Cost (\$/kg)	2.55	4.65	4.80

Example - Trail Mix

If the manufacturer wanted to use up all the ingredients in storage each day, how many batches of each mixture (airline, gas station, and grocery) should be made?

We can set this up as a system of equations, one for each ingredient, where

$$a = \text{batches of airline mixture} \quad (1)$$

$$g = \text{batches of grocery mixture} \quad (2)$$

$$s = \text{batches of gas station mixture} \quad (3)$$

$$a, g, s \geq 0 \quad (4)$$

as follows:

$$\begin{cases} 7a + 2g + 6s = 380 & \text{(Raisins)} \\ 6a + 5g + 4s = 500 & \text{(Peanuts)} \\ 2a + 8g + 5s = 620 & \text{(Chocolate)} \end{cases}$$

Example - Trail Mix

$$\begin{cases} 7a + 2g + 6s = 380 & \text{(Raisins)} \\ 6a + 5g + 4s = 500 & \text{(Peanuts)} \\ 2a + 8g + 5s = 620 & \text{(Chocolate)} \end{cases}$$

We can then transform this system of equations into matrix form:

$$\begin{pmatrix} 7 & 2 & 6 \\ 6 & 5 & 4 \\ 2 & 8 & 5 \end{pmatrix} \begin{pmatrix} a \\ g \\ s \end{pmatrix} = \begin{pmatrix} 380 \\ 500 \\ 620 \end{pmatrix}$$

While we haven't yet discussed how to solve such a system of equations, you can verify that

$$a = 20 \text{ batches} \quad g = 60 \text{ batches} \quad s = 20 \text{ batches}$$

is indeed a solution.

Side note: Matrix Inverse

Once we transform the system of equations into matrix form:

$$\mathbf{Ax} = \mathbf{b}$$

, A good question to ask is this: Does the matrix \mathbf{A} have an inverse? If so, we could multiply both sides by the inverse of \mathbf{A} :

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Thus, if we know the inverse of \mathbf{A} we can solve this problem by simple multiplication.

Unfortunately, we don't often deal with situations where inverse matrices exist!

Example - Trail Mix

Use matrix-vector multiplication to determine how much it costs the manufacturer to produce 1 batch of each mixture.

Knowing the prices of individual ingredients makes this easy.
We simply need to compute the linear combination

$$\$2.55 \star (\text{kg of Raisins}) + \$4.65 \star (\text{kg of Peanuts}) + \$4.80 \star (\text{kg of Chocolate})$$

For each type of mixture.

Example - Trail Mix

The table contains the information about the mixes, each row containing the recipe for a 15 kg batch.

	Raisins (kg/batch)	Peanuts (kg/batch)	Chocolate (kg/batch)
Airline (a)	7	6	2
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Example - Trail Mix

We simply need to compute the linear combination

$$\$2.55 \star (\text{kg of Raisins}) + \$4.65 \star (\text{kg of Peanuts}) + \$4.80 \star (\text{kg of Chocolate})$$

As we've seen in the last lecture, this can be done with the matrix-vector product:

$$\begin{pmatrix} 7 & 6 & 2 \\ 2 & 5 & 8 \\ 6 & 4 & 5 \end{pmatrix} \begin{pmatrix} 2.55 \\ 4.65 \\ 4.80 \end{pmatrix} = \begin{pmatrix} 55.35 \\ 66.75 \\ 57.90 \end{pmatrix}$$

Example - Trail Mix

$$\begin{pmatrix} 7 & 6 & 2 \\ 2 & 5 & 8 \\ 6 & 4 & 5 \end{pmatrix} \begin{pmatrix} 2.55 \\ 4.65 \\ 4.80 \end{pmatrix} = \begin{pmatrix} 55.35 \\ 66.75 \\ 57.90 \end{pmatrix}$$

Thus, the cost of 1 batch of each type of mixture is

Mixture	Cost (\$/batch)
airline	55.35
grocery	66.75
gas station	57.90

Big Systems of Equations

When we consider matrix-matrix products, the systems that we can generate get very large very quickly.

Take for example a 2×2 example

$$\mathbf{A}\mathbf{X} = \mathbf{B}$$

where

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \quad \mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 5 & 9 \end{pmatrix}$$

Each element in the right-hand side matrix is associated with an equation. For example, the first row of \mathbf{A} times the first column of \mathbf{X} gives us \mathbf{B}_{11} :

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = 2x_{11} + 3x_{21} = 7$$

Big Systems of Equations

We have one equation for each element of **B**! Let's make sure we understand how to get all 4 of these equations:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \quad \mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 5 & 9 \end{pmatrix}$$

$$2x_{11} + 3x_{21} = 7$$

$$2x_{12} + 3x_{22} = 6$$

$$1x_{11} + 4x_{21} = 5$$

$$1x_{12} + 4x_{22} = 9$$

Regression Analysis as a System of Equations

Consider a regression analysis with some dependent variable, \mathbf{y} , and two independent variables, \mathbf{h} , \mathbf{w} .

The preliminary goal of this analysis is to find unknown parameters β_0 , β_1 , and β_2 such that

$$\mathbf{y} = \beta_0 + \beta_1 \mathbf{h} + \beta_2 \mathbf{w}$$

Regression Analysis as a System of Equations

This single equation makes no reference to all of our data points! Suppose, for simplicity, we have only 4 observations as listed in the following table:

h	w	y
3	3	6
2	3	6
5	6	10
6	5	9

Regression Analysis as a System of Equations

When we write the model

$$\mathbf{y} = \beta_0 + \beta_1 \mathbf{h} + \beta_2 \mathbf{w}$$

what we are really saying is that the equation holds true for each of the 4 observations in our dataset. So rather than 1 single equation, what we really have here is 4 equations - 1 for each observation:

$$\begin{cases} \beta_0 + 3\beta_1 + 3\beta_2 = 6 & \text{(obs. 1)} \\ \beta_0 + 2\beta_1 + 3\beta_2 = 6 & \text{(obs. 2)} \\ \beta_0 + 5\beta_1 + 6\beta_2 = 10 & \text{(obs. 3)} \\ \beta_0 + 6\beta_1 + 5\beta_2 = 9 & \text{(obs. 4)} \end{cases}$$

Regression Analysis as a System of Equations

Rather than writing all these equations out, we instead represent the situation in matrix format as

$$\mathbf{X}\boldsymbol{\beta} = \mathbf{y},$$

Where

$$\mathbf{X} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 2 & 3 \\ 1 & 5 & 6 \\ 1 & 6 & 5 \end{pmatrix} \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 6 \\ 6 \\ 10 \\ 9 \end{pmatrix}$$

Regression Analysis as a System of Equations

We know from our experiences with data that this situation will not have an exact solution: our data does not fall exactly on some straight line or surface.

Instead, we have to consider some error, ϵ and try to minimize it:

$$\mathbf{X}\beta + \epsilon = \mathbf{y},$$

where

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix}$$

is a vector containing the *residuals*.