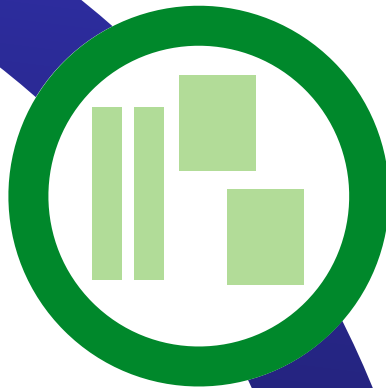


Matrix Arithmetic

(Multidimensional Math)

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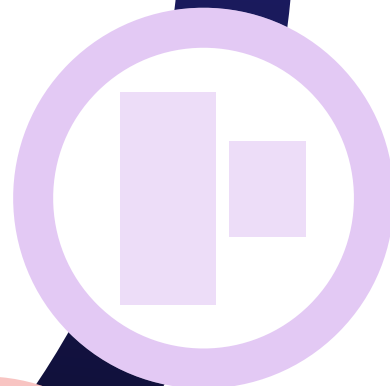
Element-wise Operations

Linear Combinations of Matrices and Vectors.



Vector Multiplication

Inner products and Matrix-Vector Multiplication



Matrix Multiplication

Inner product and linear combination viewpoint



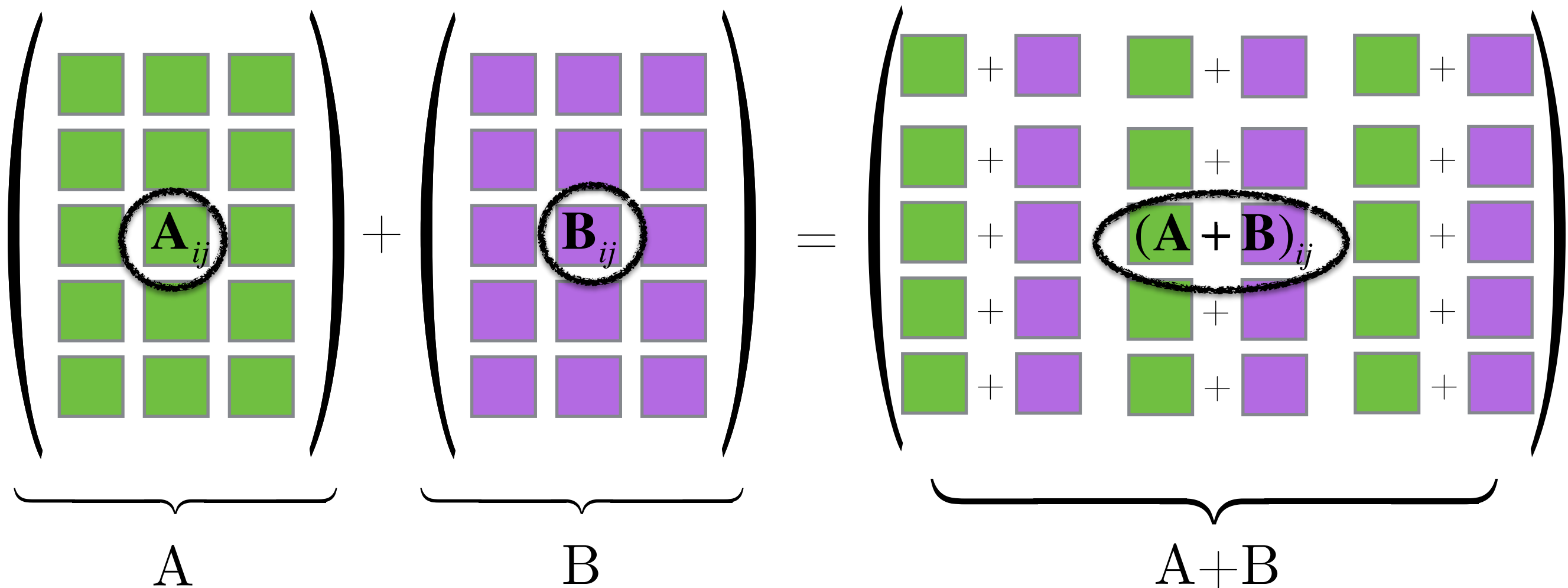
Vector Multiplication

The Outer Product

Matrix Addition/Subtraction

- ▶ Two matrices/vectors can be added/subtracted if and only if they have the same size
- ▶ Then simply add/subtract corresponding elements

Matrix Addition/Subtraction



$$(A+B)_{ij} = A_{ij} + B_{ij}$$

(Element-wise)

Example: Matrix Addition/Subtraction

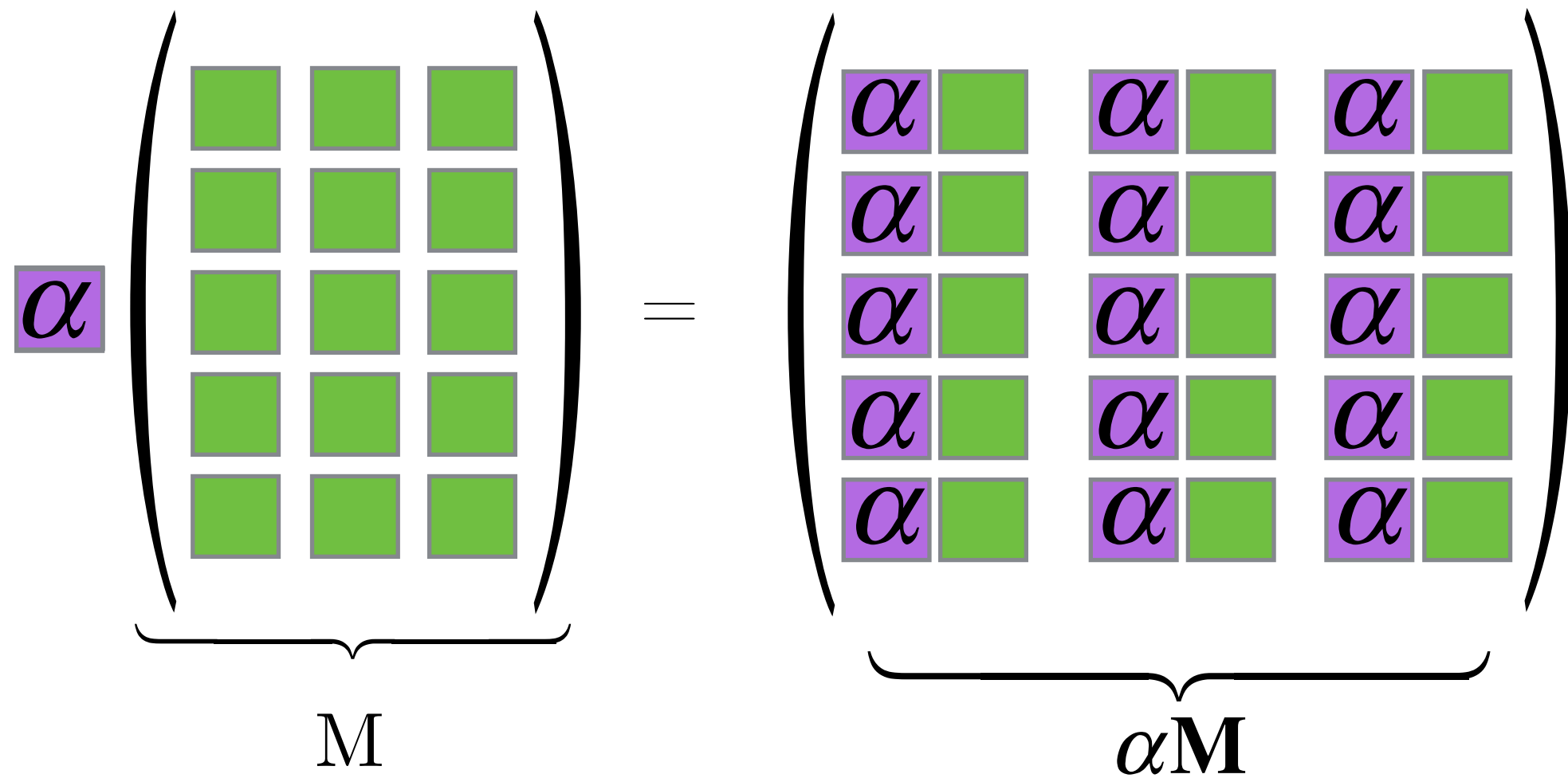
a) Compute $\mathbf{A} + \mathbf{B}$, if possible:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 4 & 5 & 6 \\ -1 & 0 & 4 \\ 3 & 4 & 3 \end{pmatrix}$$

b) Compute $\mathbf{A} - \mathbf{H}$, if possible:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \quad \mathbf{H} = \begin{pmatrix} 6 & 5 & 10 \\ 0.1 & 0.5 & 0.9 \end{pmatrix}$$

Scalar Multiplication



$$(\alpha\mathbf{M})_{ij} = \alpha\mathbf{M}_{ij}$$

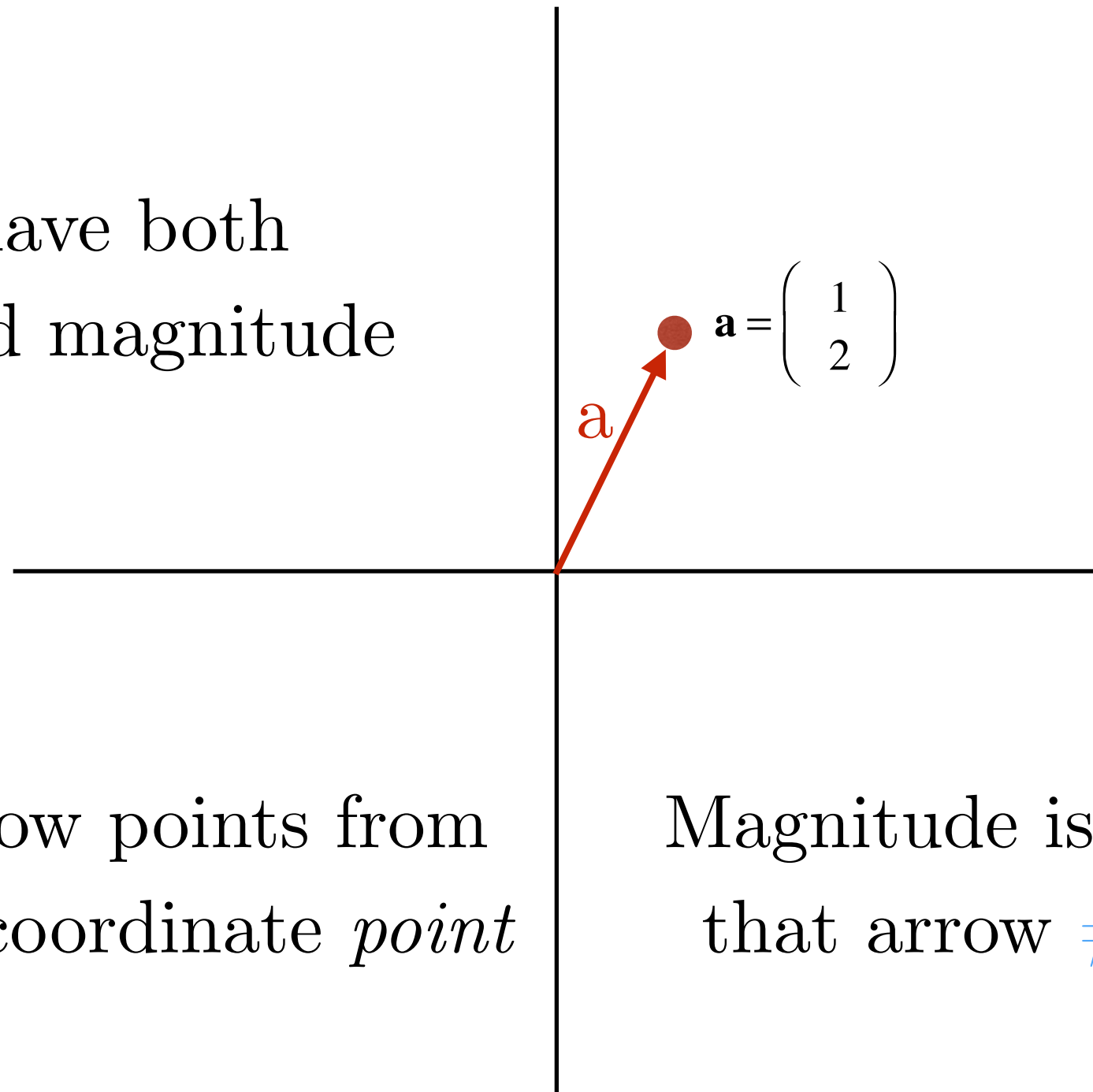
(Element-wise)

Geometric Look

Vector addition and scalar multiplication

Points \longleftrightarrow Vectors

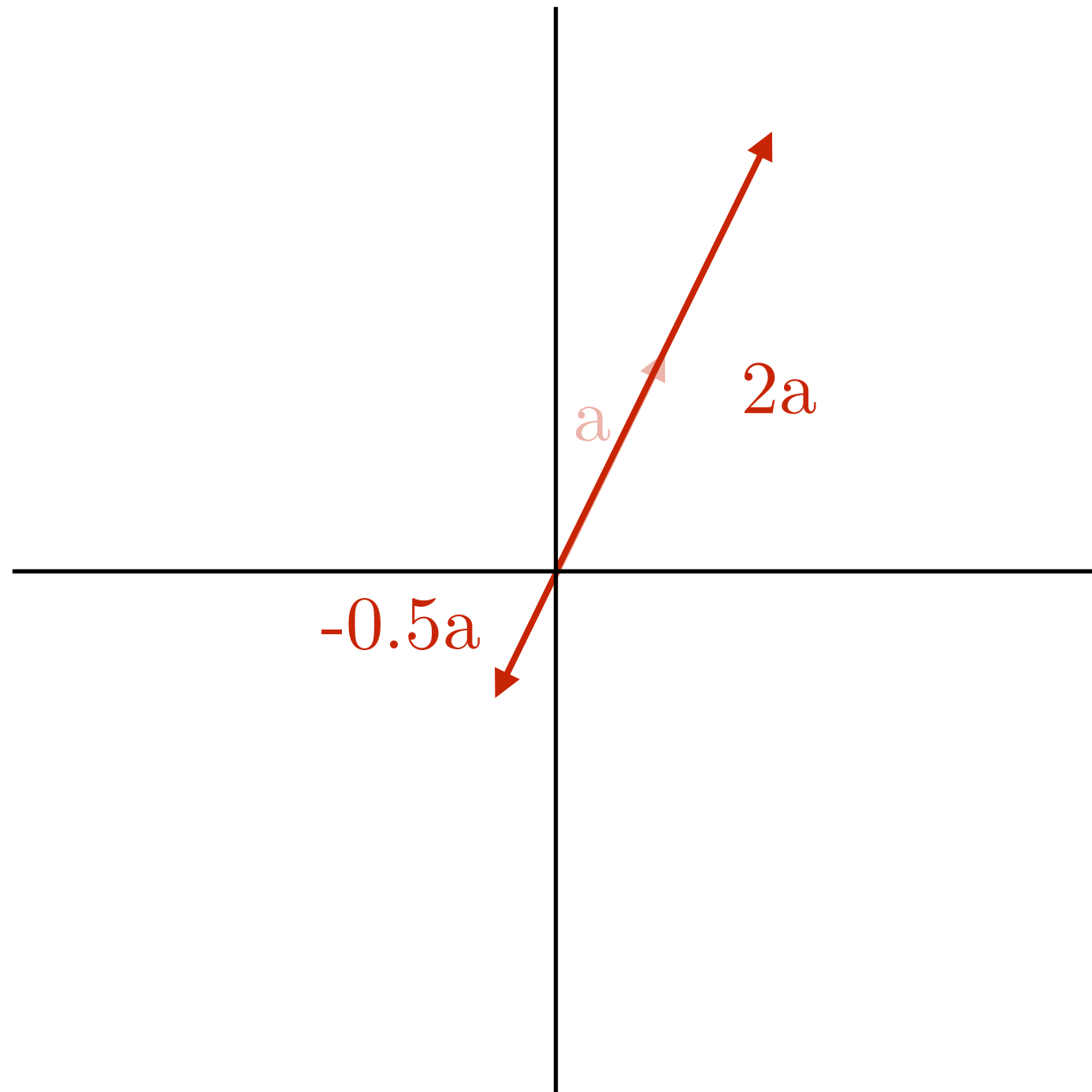
Vectors have both
direction and magnitude



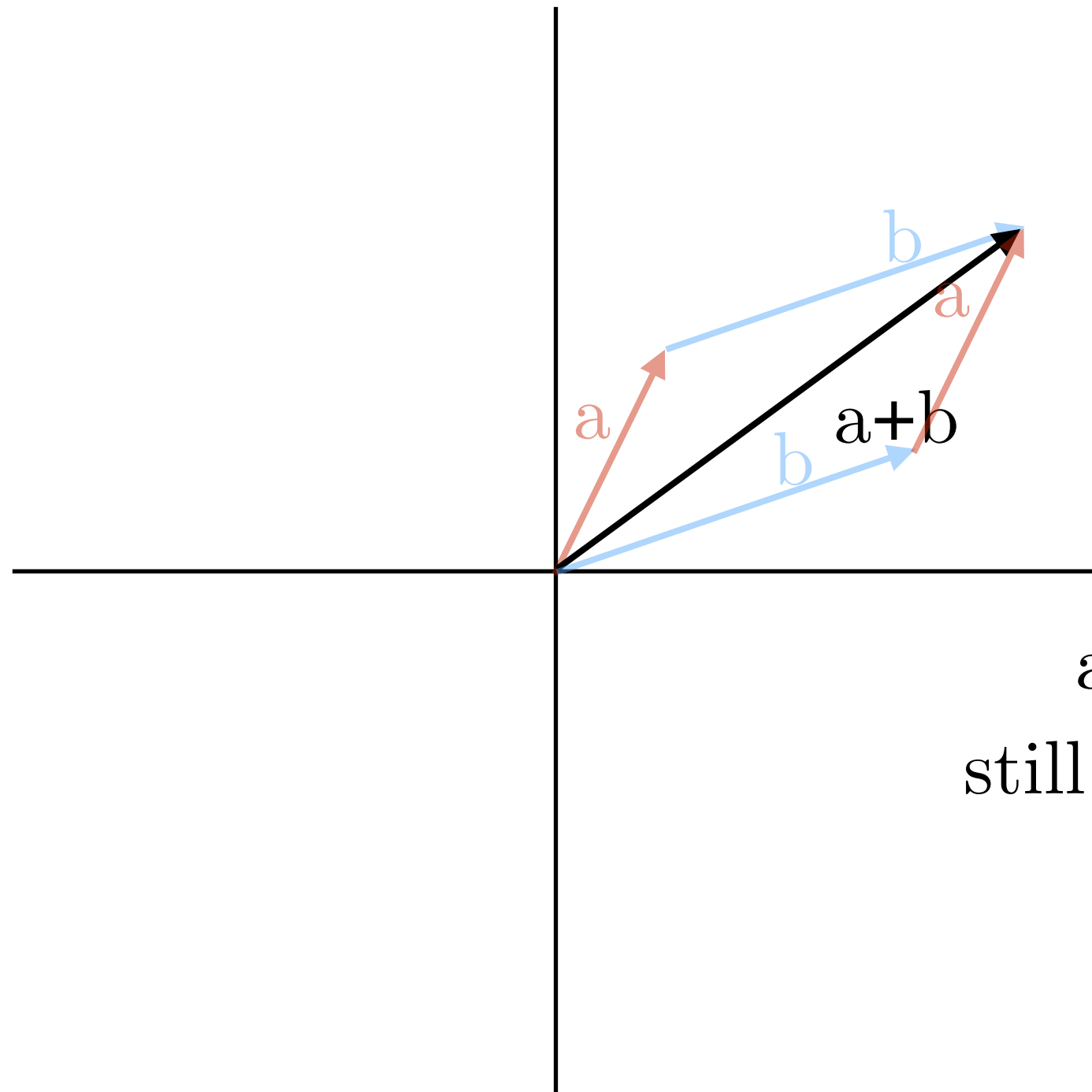
Direction arrow points from
origin to the coordinate *point*

Magnitude is the length of
that arrow [#pythagoras](#)

Scalar Multiplication (Geometrically)

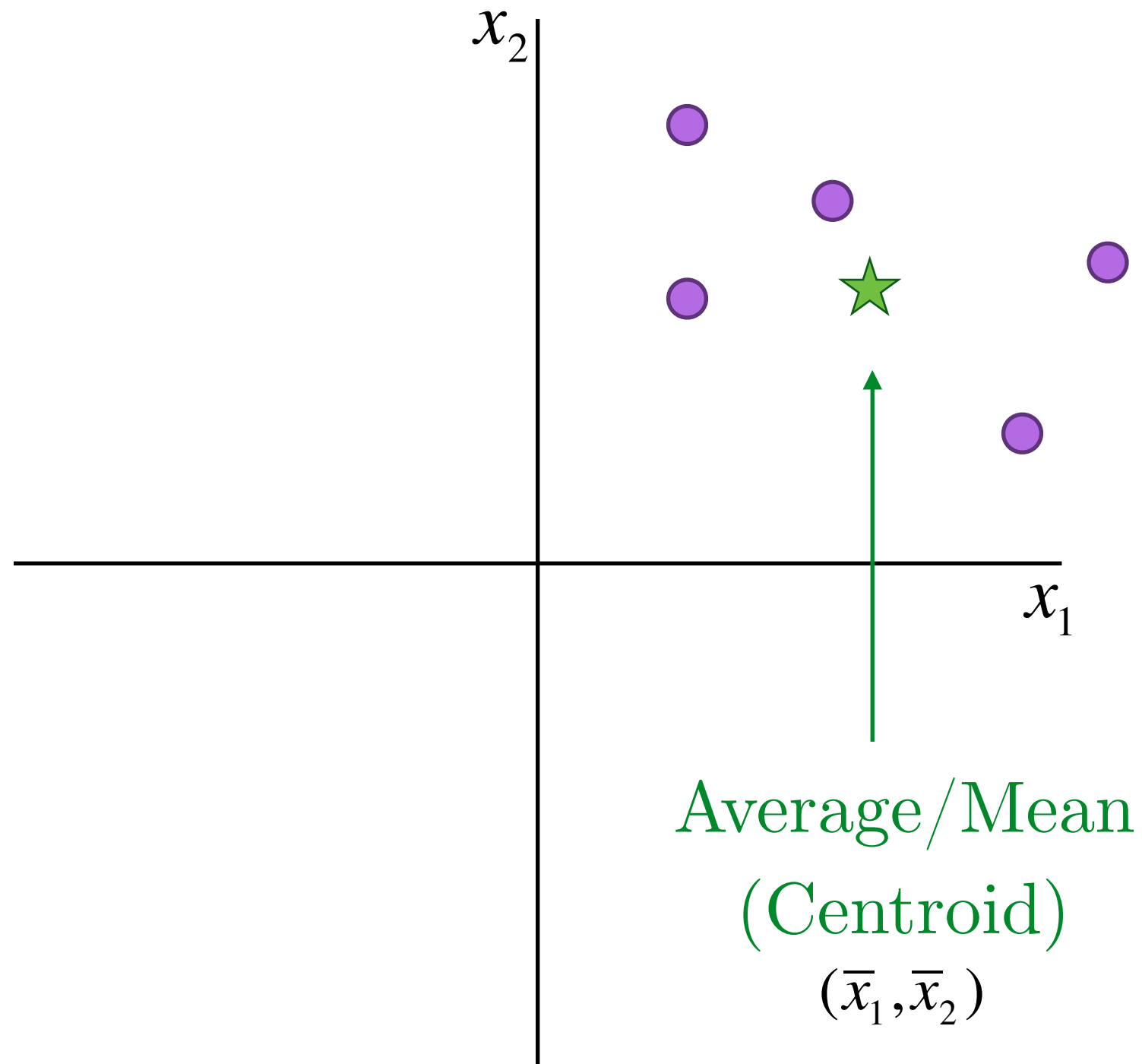


Vector Addition (Geometrically)

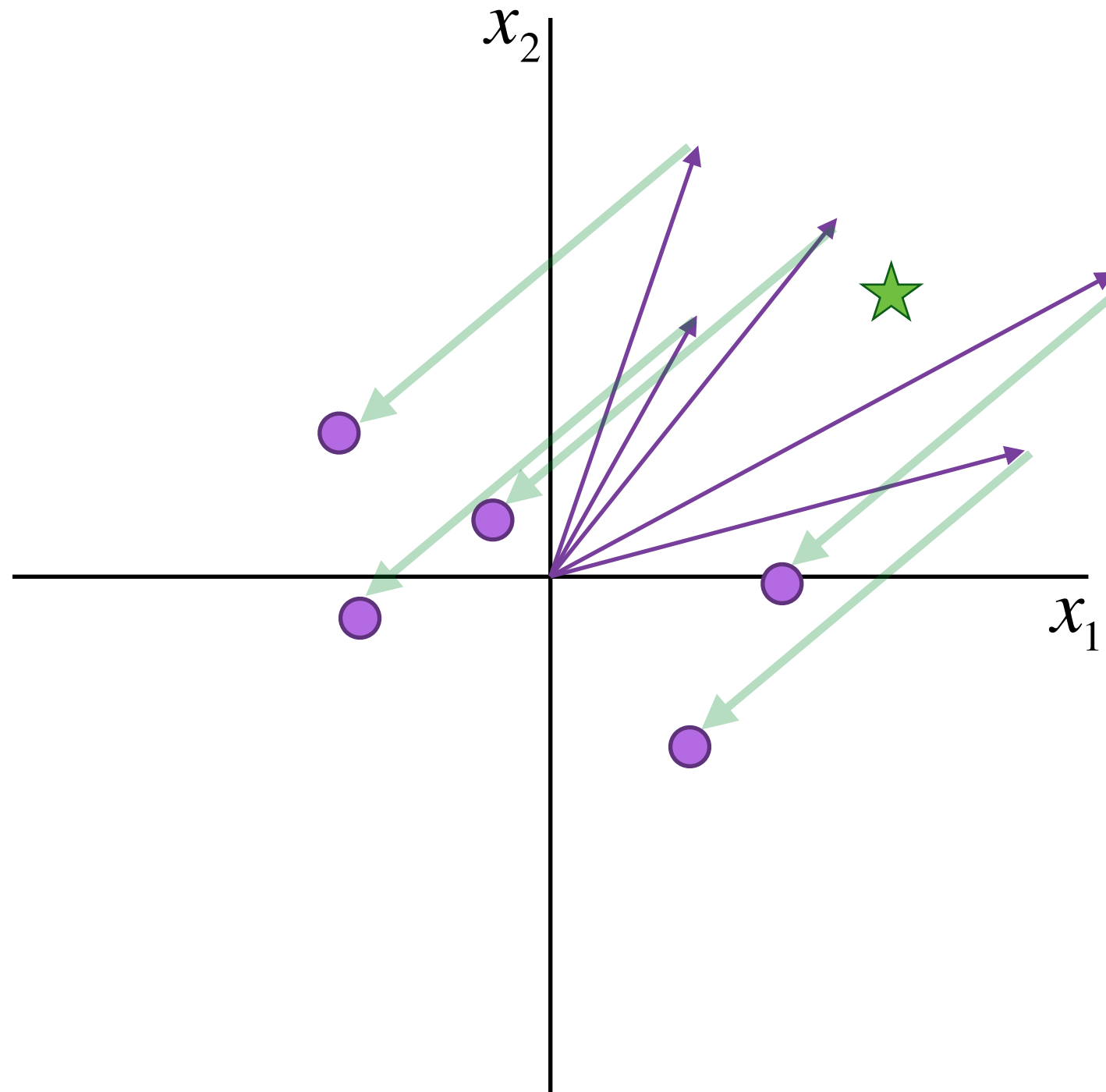


addition is
still commutative

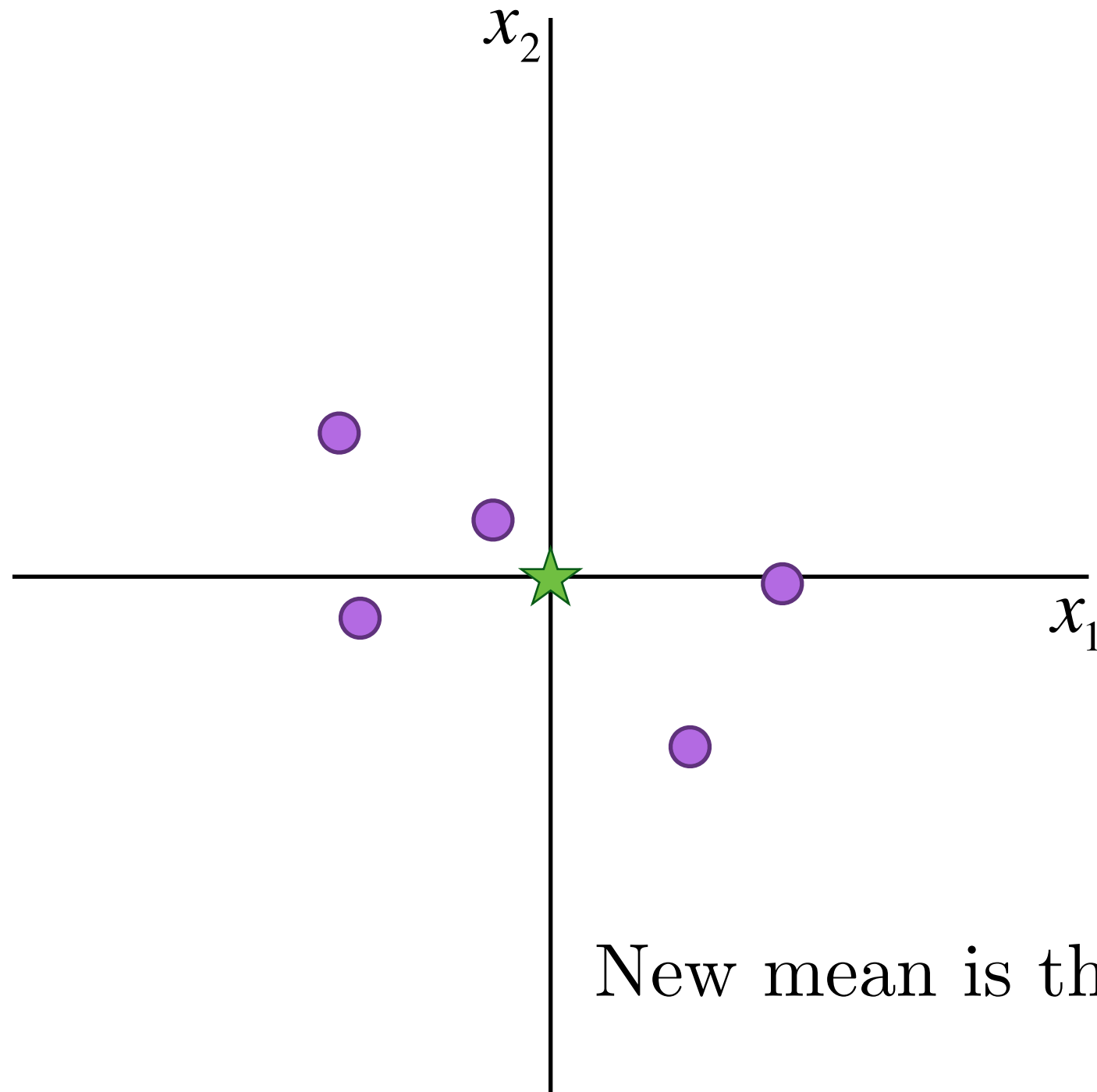
Example: Centering the data



Example: Centering the data



Example: Centering the data



New mean is the origin $(0,0)$

Linear Combinations

Linear Combinations

A linear combination of vectors is a just weighted sum:

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_p \mathbf{v}_p$$

The diagram illustrates the linear combination formula $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_p \mathbf{v}_p$. It features two sets of arrows originating from a common point at the bottom left. Three green arrows point to the scalar coefficients α_1 , α_2 , and α_p in the formula. Three purple arrows point to the vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_p in the formula. The labels 'Scalar Coefficients α_i ' and 'Vectors \mathbf{v}_i ' are placed below their respective arrow groups.

Scalar Coefficients α_i

Vectors \mathbf{v}_i

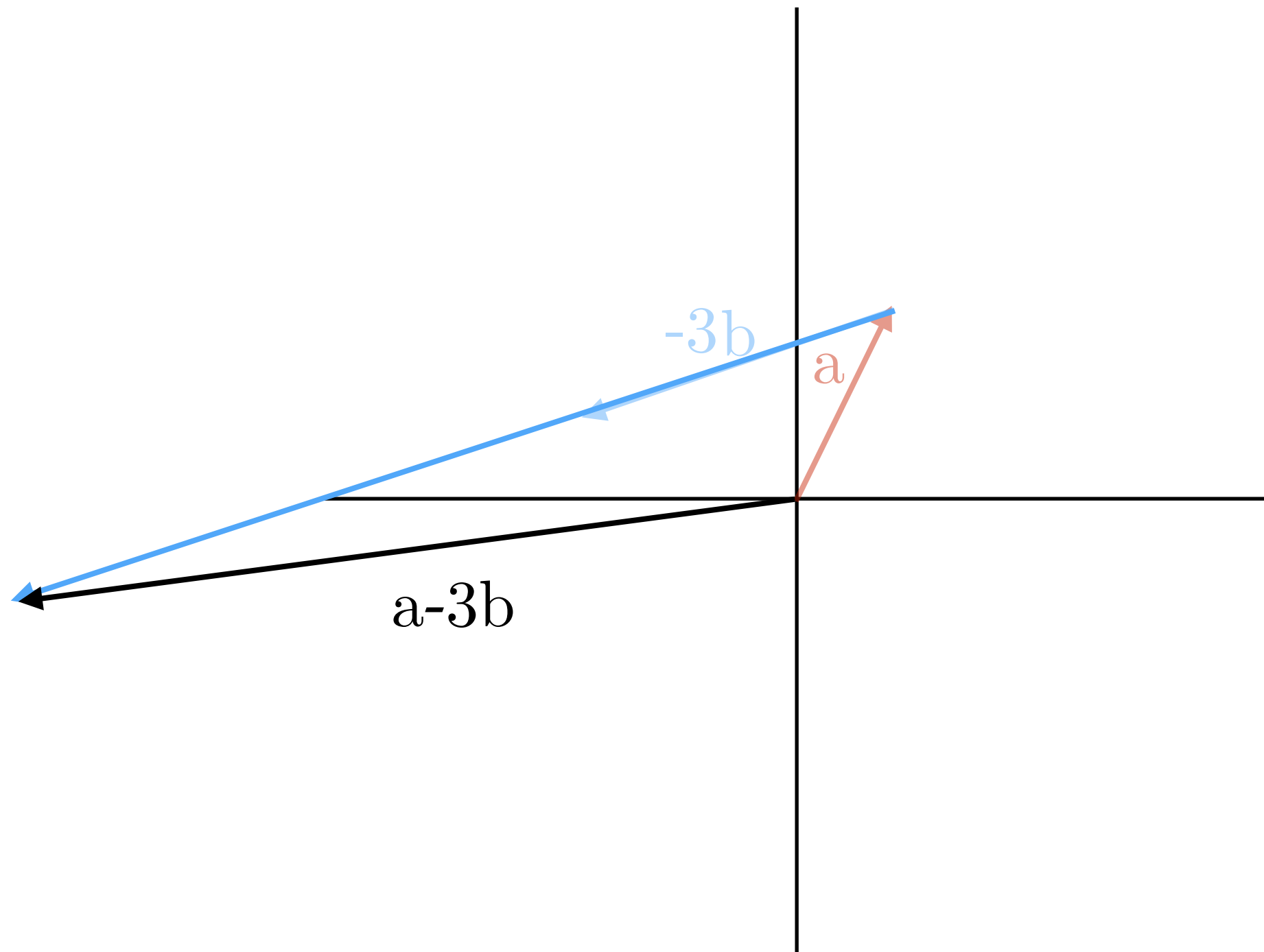
Elementary Linear Combinations

- ▶ The simplest linear combination might involve columns of the identity matrix (elementary vectors):

$$\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

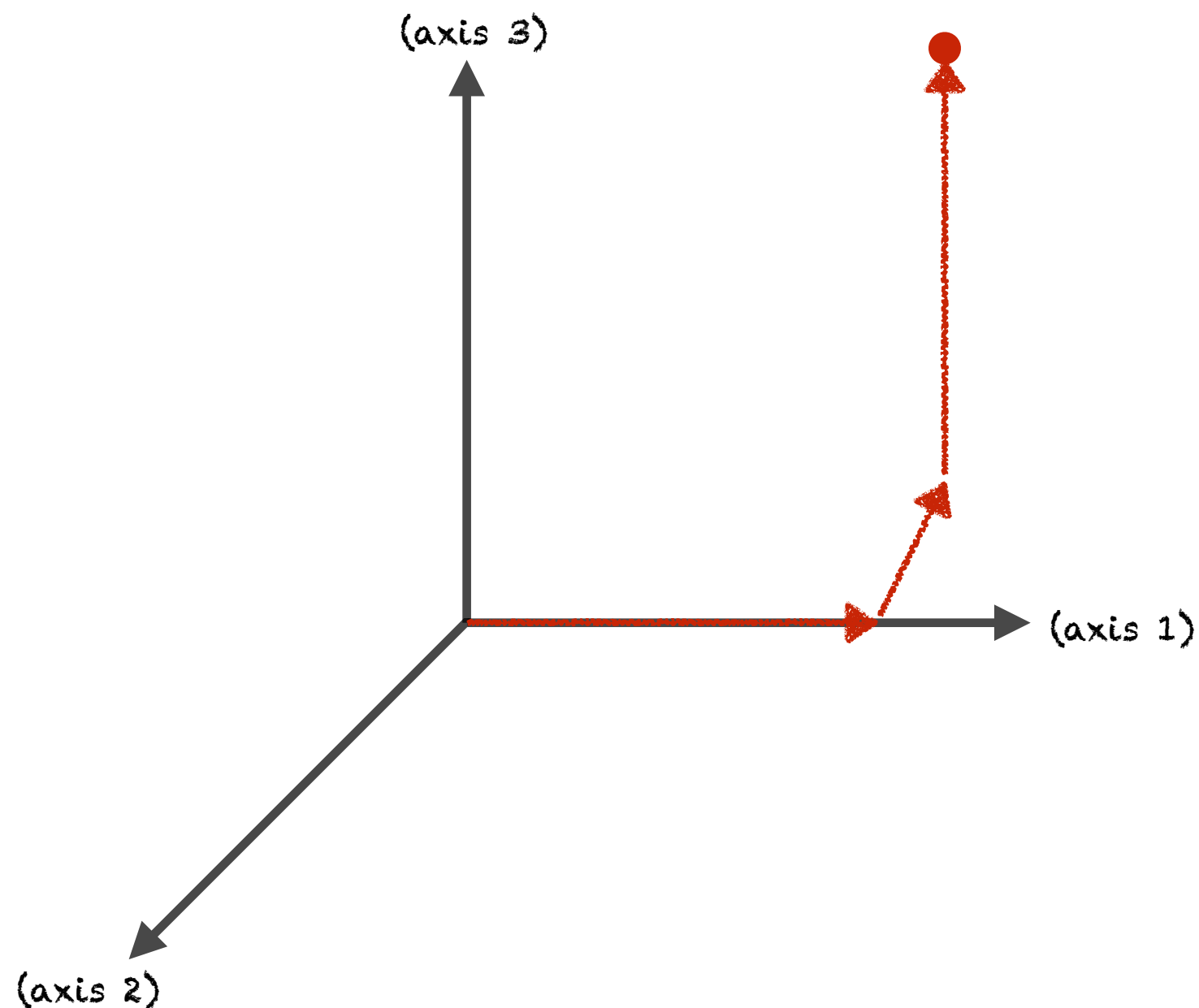
- ▶ Picture this linear combination as a “breakdown into parts” where the parts give directions along the 3 coordinate axes.

Linear Combinations (Geometrically)



Linear Combinations (Geometrically)

$$\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Example: Linear Combination of Matrices

Write the matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ as a linear combination of the following matrices:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Solution:

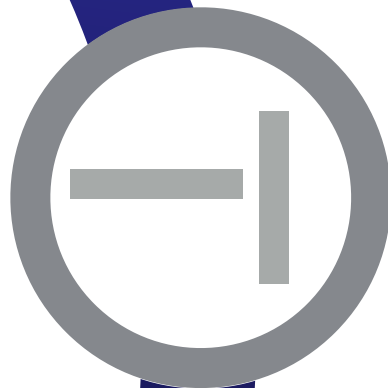
$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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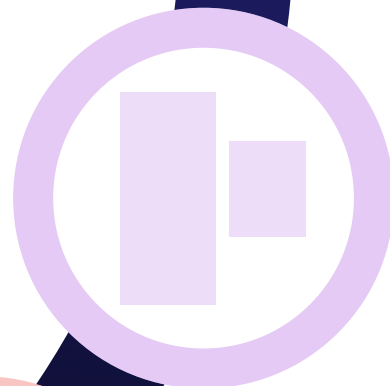
Element-wise Operations

Linear Combinations of Matrices and Vectors.



Vector Multiplication

Inner products and Matrix-Vector Multiplication



Matrix Multiplication

Inner product and linear combination viewpoint



Vector Multiplication

The Outer Product

Notation: Column vs. Row Vectors

- ▶ Throughout this course, unless otherwise specified, all vectors are assumed to be columns.
- ▶ Simplifies notation because if \mathbf{x} is a column vector:

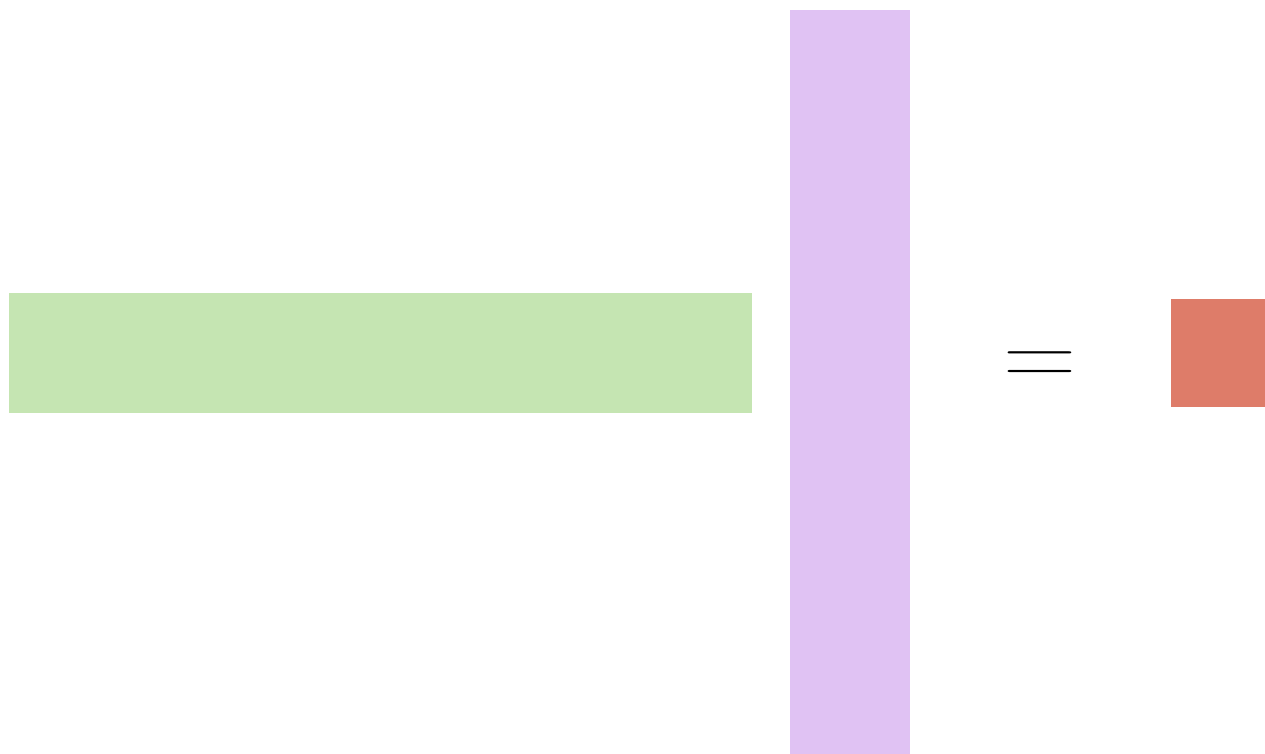
$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

then we can automatically assume that \mathbf{x}^T is a row vector:

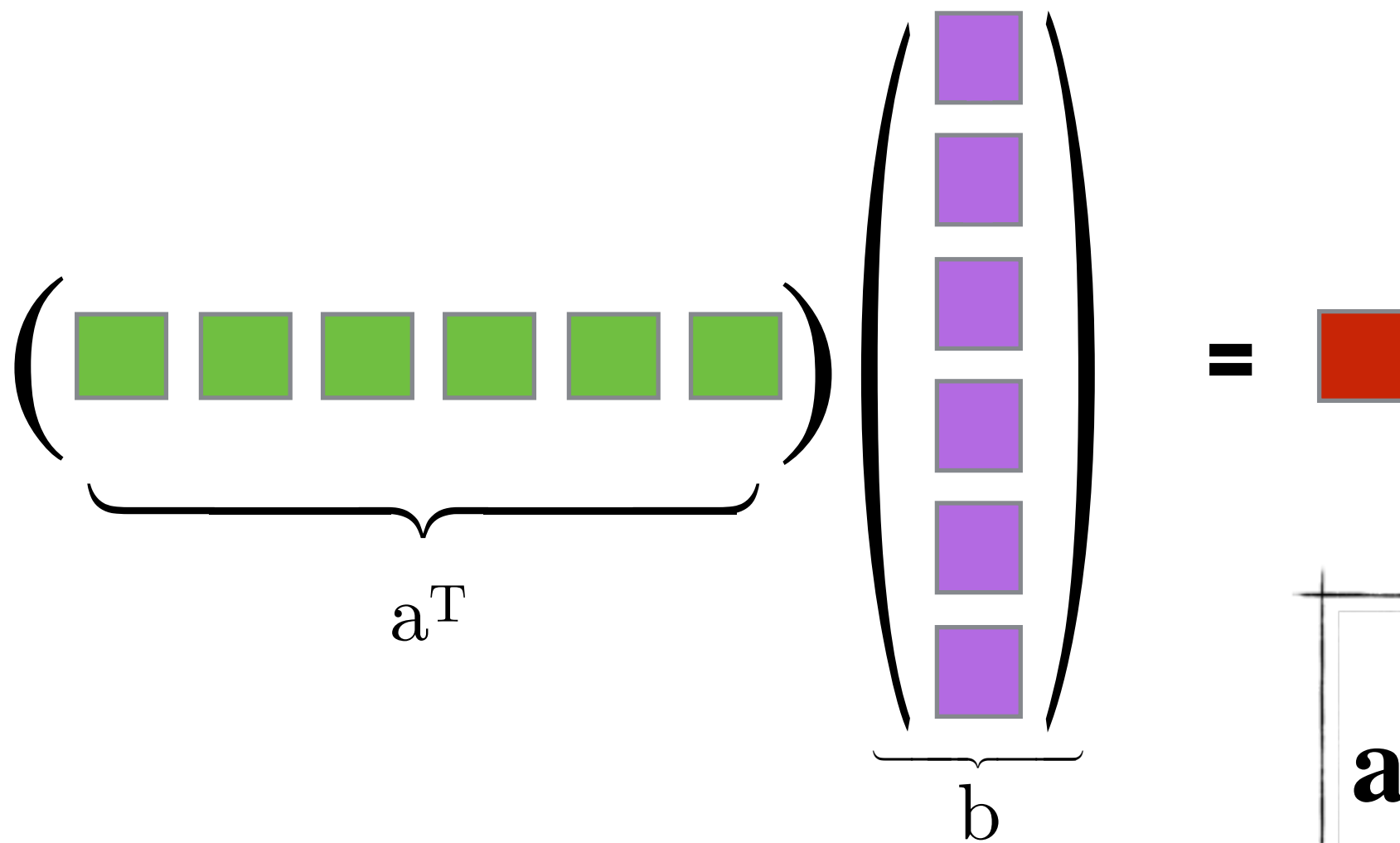
$$\mathbf{x}^T = (x_1 \ x_2 \ \dots \ x_n)$$

Vector Inner Product

- ▶ The vector inner product is the multiplication of a row vector times a column vector.
- ▶ It is known across broader sciences as the ‘dot product’.
- ▶ The result of this product is a scalar.



Inner Product (row x column)



$$\mathbf{a}^T \mathbf{b} = \sum_{i=1}^n a_i b_i$$

Inner Product (row x column)

$$\left(\begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} * \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} + \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} + \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} + \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} + \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} + \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \\ \square \end{array} \right) = \square$$

a and **b** must have the same number of elements.

Examples: Inner Product

Let

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 4 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 3 \\ 5 \\ 1 \\ 7 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} -3 \\ -2 \\ 5 \\ 3 \\ -2 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \\ -3 \\ -2 \end{pmatrix}$$

If possible, compute the following inner products:

a. $\mathbf{x}^T \mathbf{y}$

Examples: Inner Product

Let

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 4 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 3 \\ 5 \\ 1 \\ 7 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} -3 \\ -2 \\ 5 \\ 3 \\ -2 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \\ -3 \\ -2 \end{pmatrix}$$

If possible, compute the following inner products:

b. $\mathbf{x}^T \mathbf{v}$

c. $\mathbf{v}^T \mathbf{u}$

Check your Understanding

Let

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{p} = \begin{pmatrix} 0.5 \\ 0.1 \\ 0.2 \\ 0 \\ 0.2 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 10 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{s} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

If possible, compute the following inner products:

a. $\mathbf{v}^T \mathbf{e}$

d. $\mathbf{p}^T \mathbf{u}$

b. $\mathbf{e}^T \mathbf{v}$

e. $\mathbf{v}^T \mathbf{v}$

c. $\mathbf{v}^T \mathbf{s}$

Check your Understanding SOLUTION

Let

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{p} = \begin{pmatrix} 0.5 \\ 0.1 \\ 0.2 \\ 0 \\ 0.2 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 10 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{s} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

If possible, compute the following inner products:

a. $\mathbf{v}^T \mathbf{e} = 15$

d. $\mathbf{p}^T \mathbf{u} = 6.2$

b. $\mathbf{e}^T \mathbf{v} = 15$

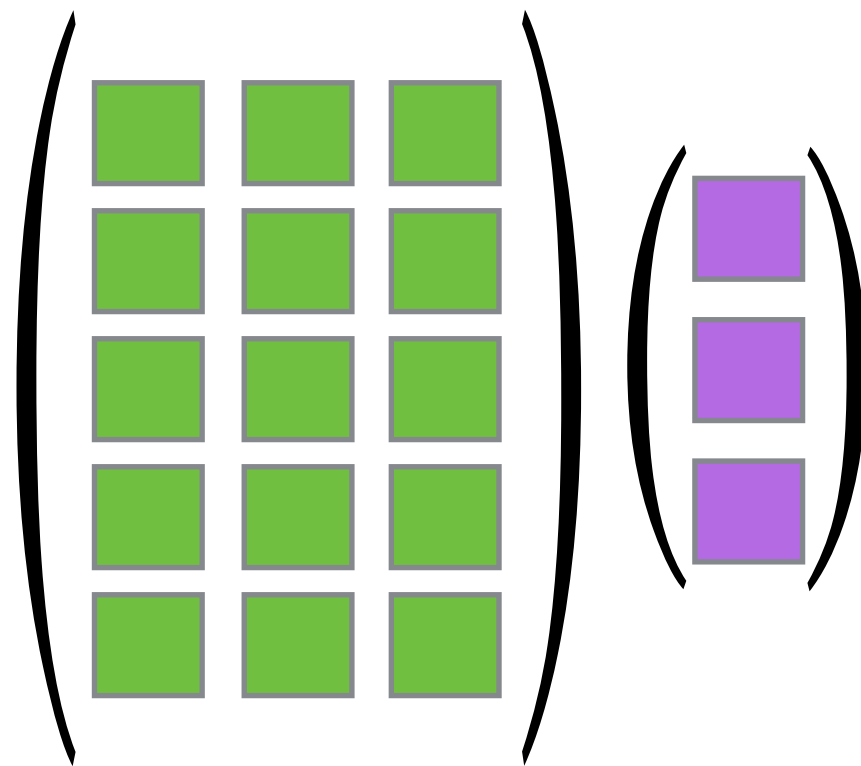
e. $\mathbf{v}^T \mathbf{v} = 55$

c. $\mathbf{v}^T \mathbf{s} = \text{not possible}$

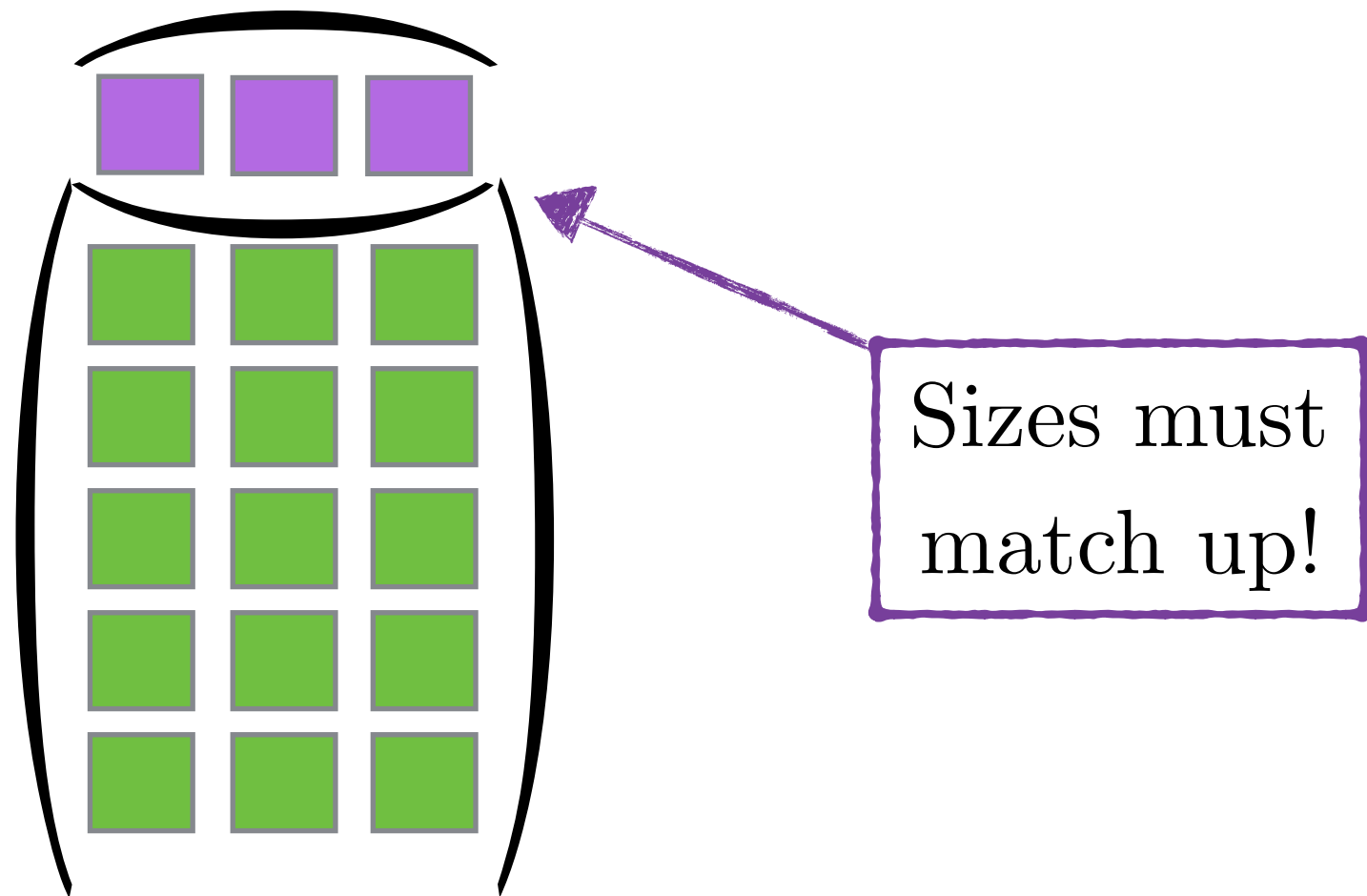
Matrix-Vector Multiplication

Inner Product View (I-P View)

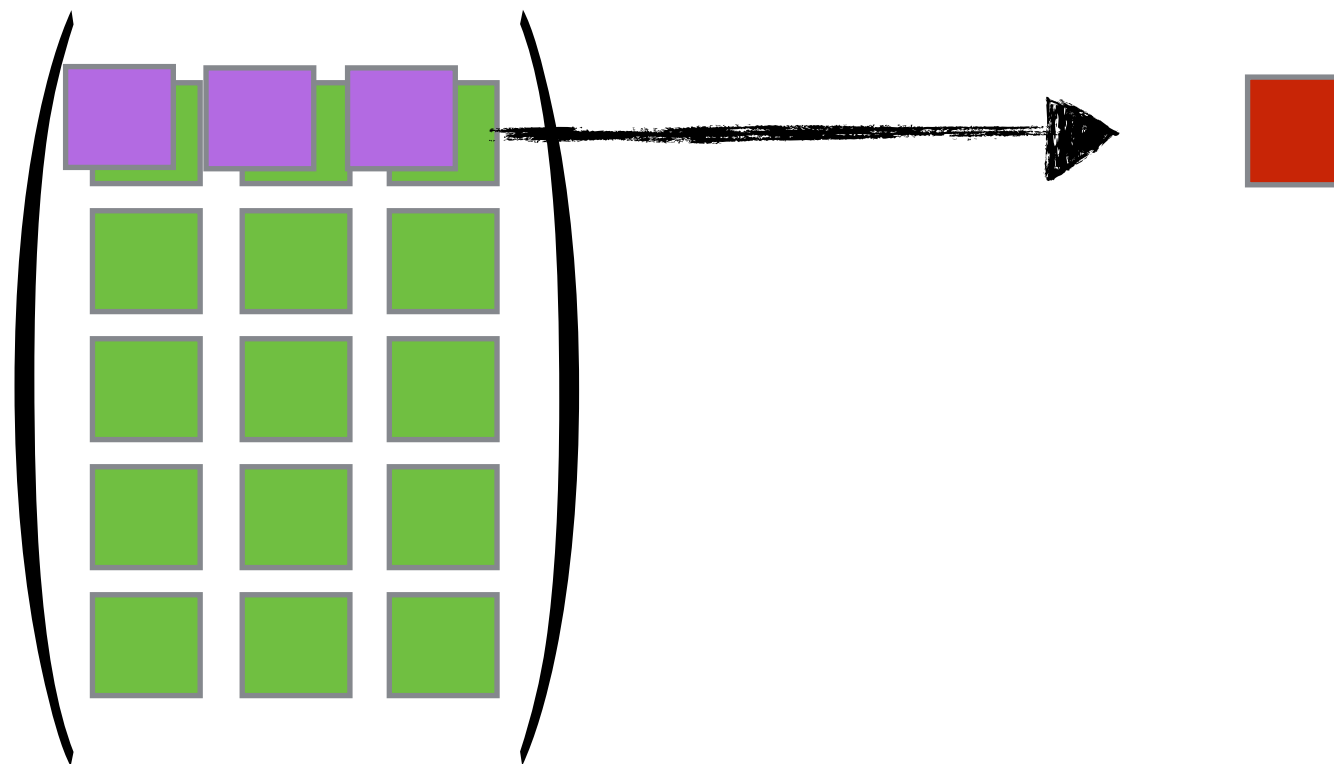
Matrix-Vector Multiplication (I-P view)



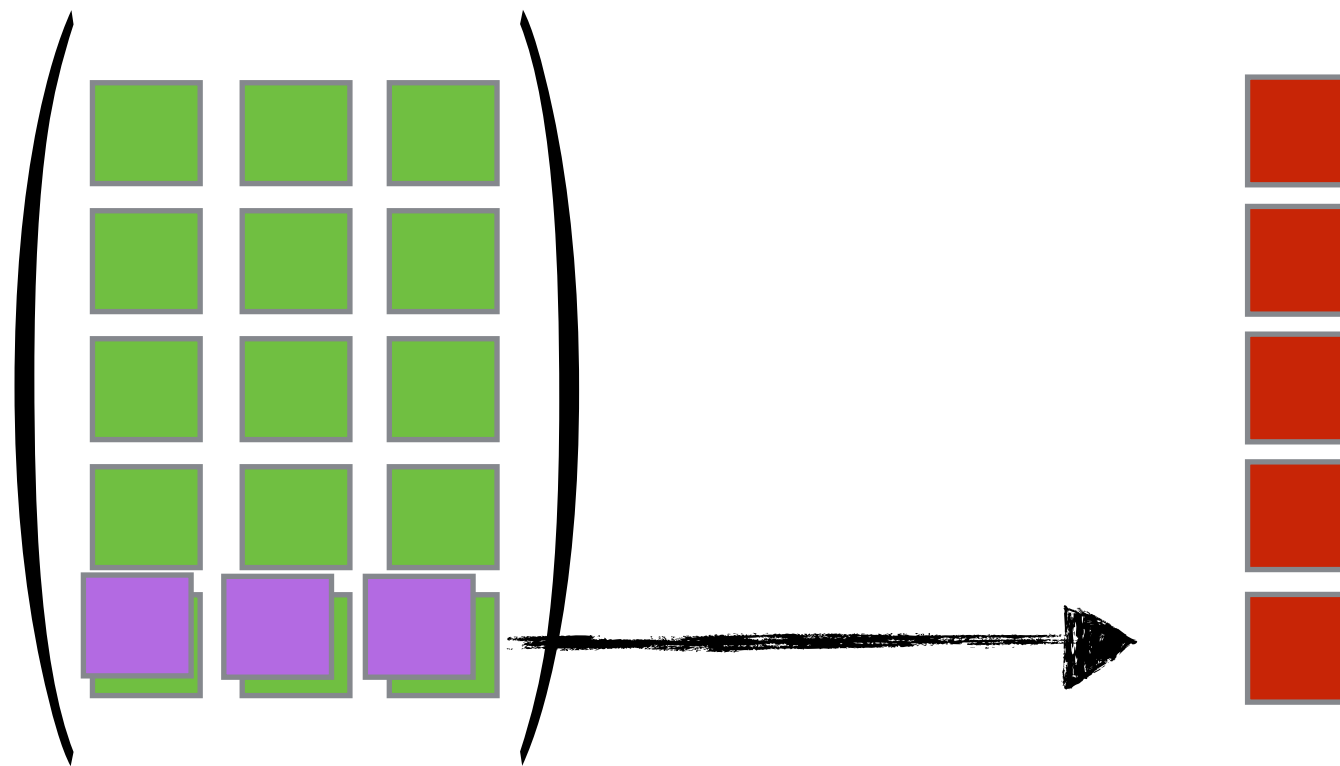
Matrix-Vector Multiplication (I-P view)



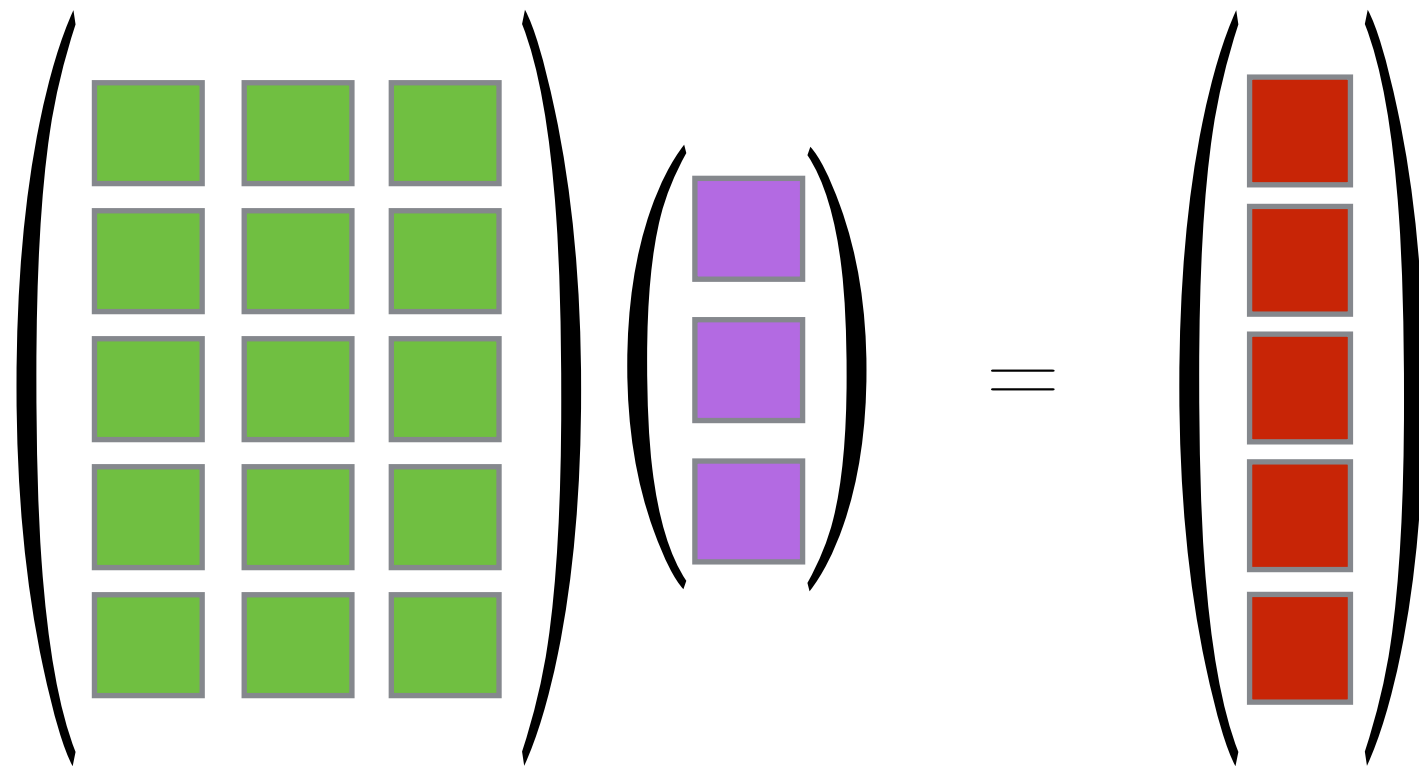
Matrix-Vector Multiplication (I-P view)



Matrix-Vector Multiplication (I-P view)



Matrix-Vector Multiplication (I-P view)



Example: Matrix-Vector Products

Let

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \mathbf{q} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Determine whether the following matrix-vector products are possible. When possible, compute the product.

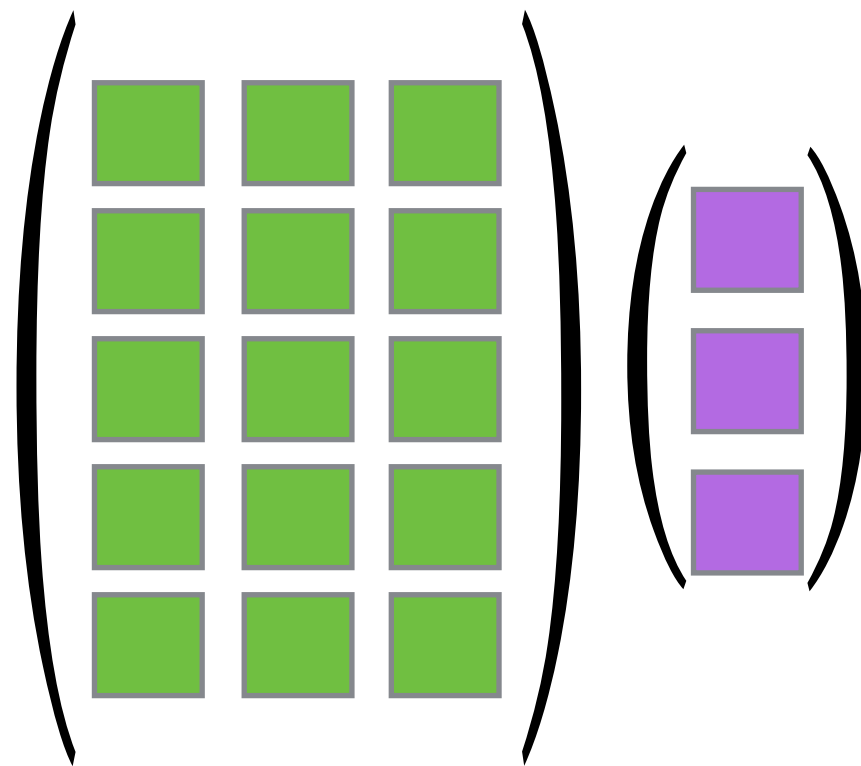
a. \mathbf{Aq}

b. \mathbf{Av}

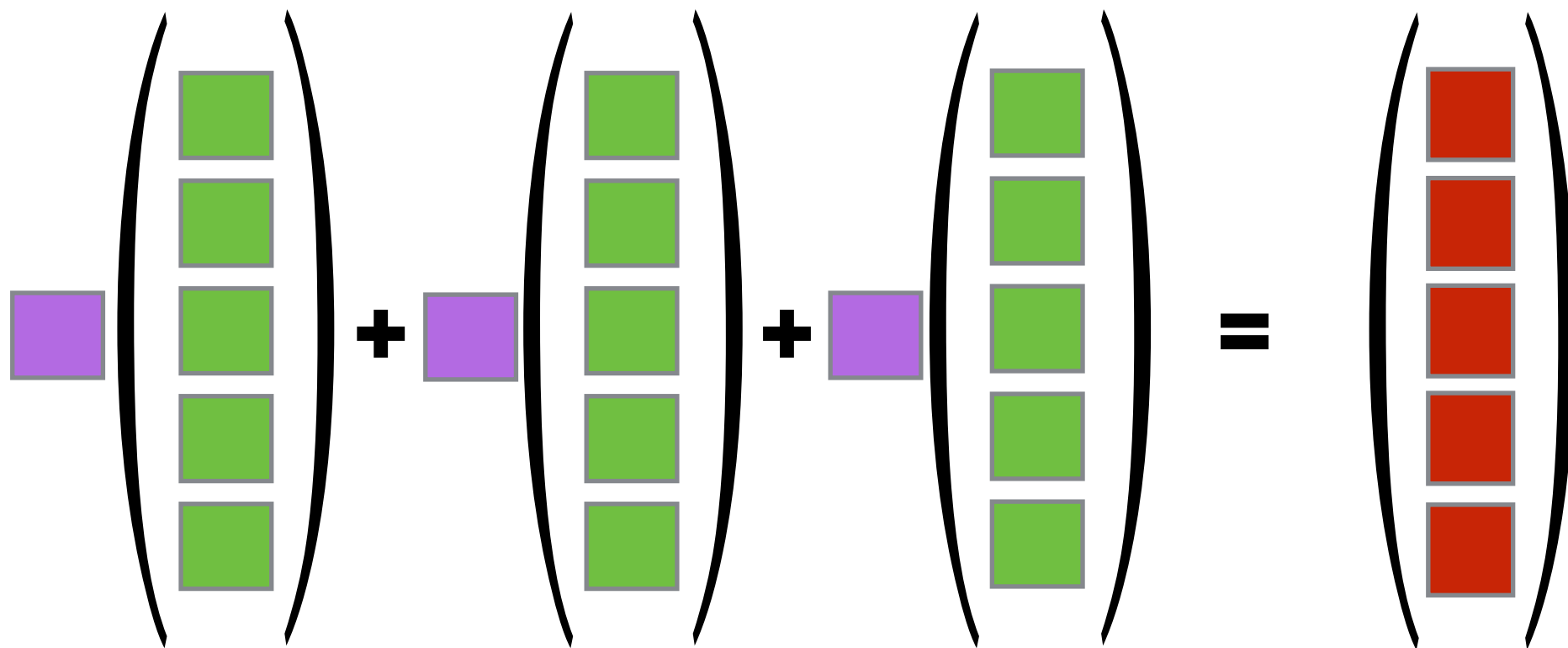
Matrix-Vector Multiplication

Linear Combination View (L-C View)

Matrix-Vector Multiplication (L-C view)



Matrix-Vector Multiplication (L-C view)



Example: Linear Combination View

$$\mathbf{Av} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2(\textcolor{red}{3}) + 3(\textcolor{blue}{2}) \\ -1(\textcolor{red}{3}) + 4(\textcolor{blue}{2}) \\ 5(\textcolor{red}{3}) + 1(\textcolor{blue}{2}) \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} \textcolor{red}{3} \\ \textcolor{blue}{2} \end{pmatrix}$$

Example: Linear Combination View

$$\mathbf{A}\mathbf{v} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2(\textcolor{red}{3}) + 3(\textcolor{blue}{2}) \\ -1(\textcolor{red}{3}) + 4(\textcolor{blue}{2}) \\ 5(\textcolor{red}{3}) + 1(\textcolor{blue}{2}) \end{pmatrix}$$

$$= \textcolor{red}{3} \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \textcolor{blue}{2} \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

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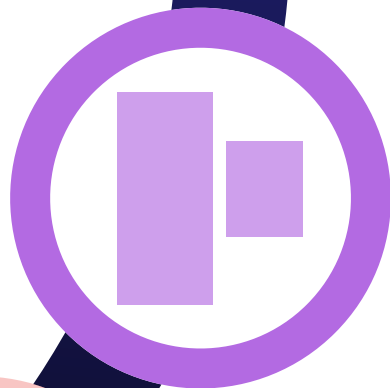
Element-wise Operations

Linear Combinations of Matrices and Vectors.



Vector Multiplication

Inner products and Matrix-Vector Multiplication



Matrix Multiplication

Inner product and linear combination viewpoint



Vector Multiplication

The Outer Product

Matrix-Matrix Multiplication

- Matrix multiplication is **NOT** commutative.

$$AB \neq BA$$

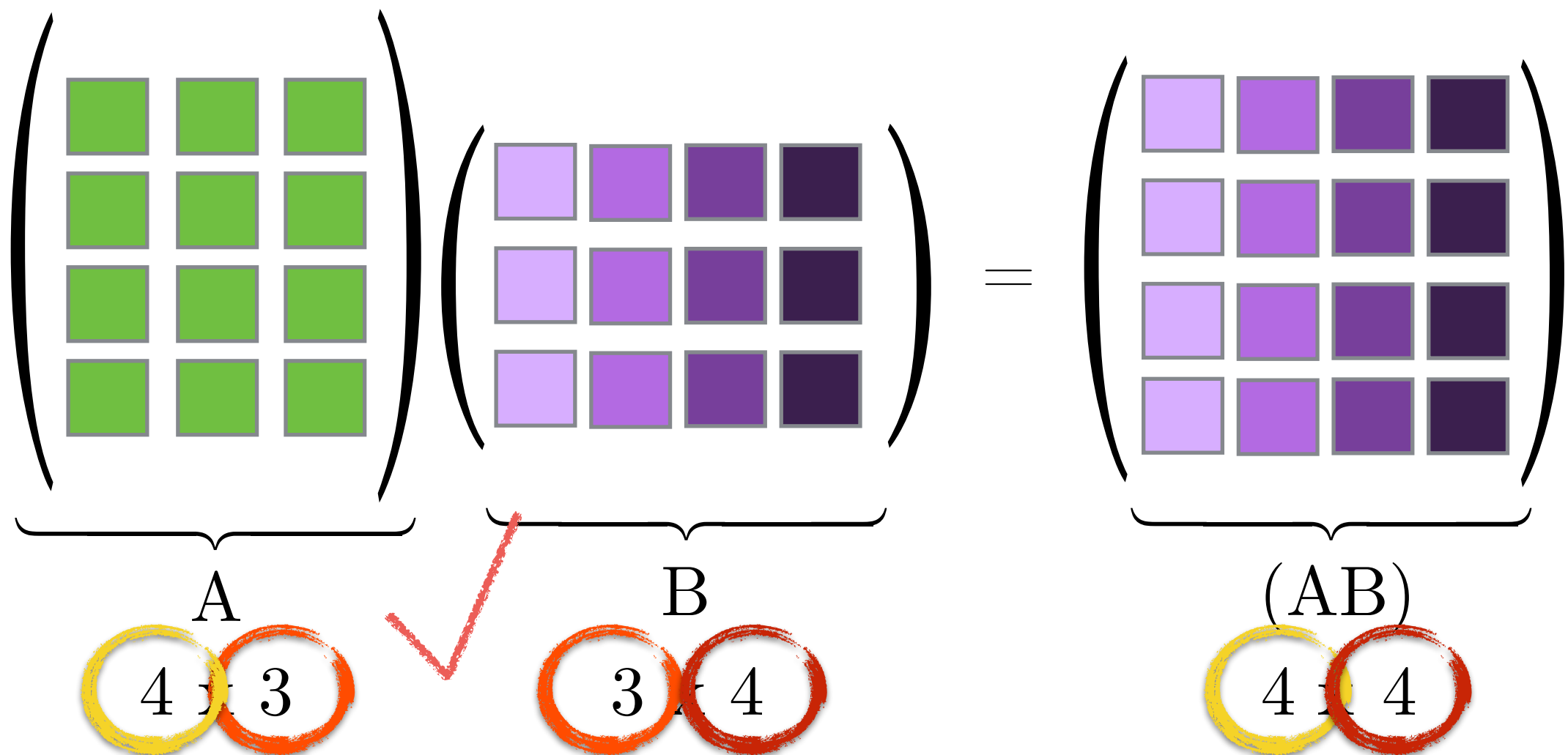
- Matrix multiplication is only defined for dimension-compatible matrices

Matrix-Matrix Multiplication (I-P View)

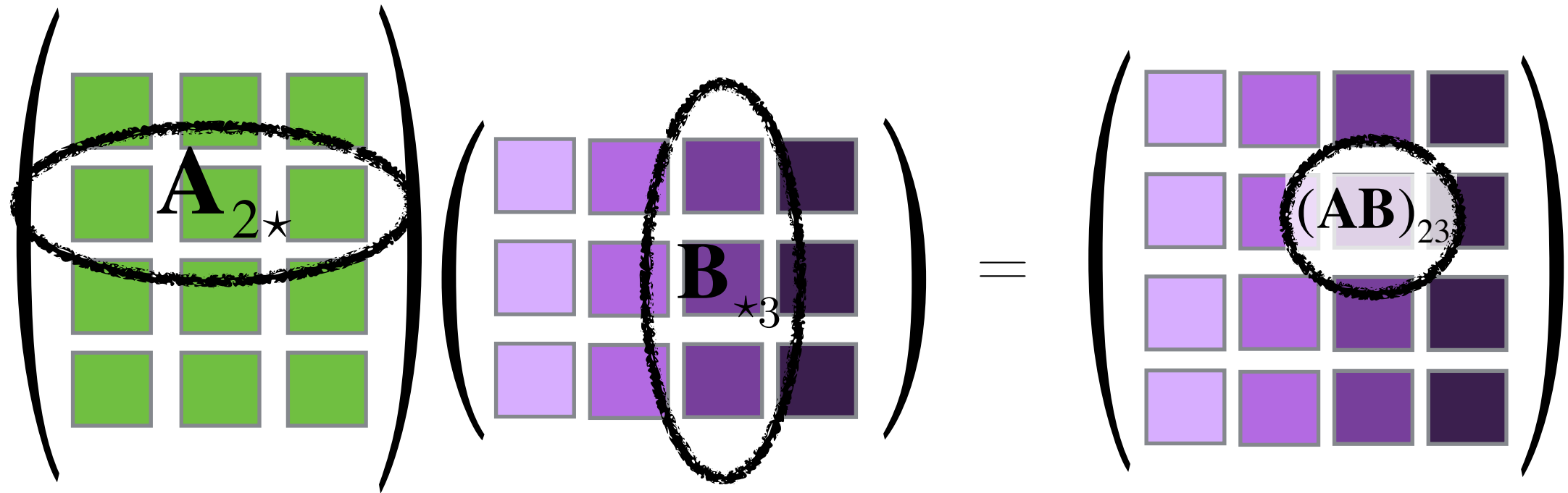
- If A and B are dimension compatible, then we compute the product AB by multiplying every row of A by every column of B (*inner products*).
- The $(i,j)^{\text{th}}$ entry of the product AB is the i^{th} row of A multiplied by the j^{th} column of B

Matrix-Matrix Multiplication (I-P View)

A and B are *dimension compatible* for the product AB if the number of columns in A is equal to the number of rows in B



Matrix-Matrix Multiplication (I-P View)



$$(\mathbf{AB})_{ij} = \mathbf{A}_{i\star} \mathbf{B}_{\star j}$$

Example: Matrix-Matrix Multiplication

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & -2 \\ 2 & -3 \end{pmatrix}$$

Check Your Understanding

Suppose we have

$$\mathbf{A}_{4 \times 6} \quad \mathbf{B}_{5 \times 5} \quad \mathbf{M}_{5 \times 4} \quad \mathbf{P}_{6 \times 5}$$

Circle the matrix products that are possible to compute and write the dimension of the result.

$$\mathbf{AM} \quad \mathbf{MA} \quad \mathbf{BM} \quad \mathbf{MB} \quad \mathbf{PA} \quad \mathbf{PM} \quad \mathbf{AP} \quad \mathbf{A}^T \mathbf{P} \quad \mathbf{M}^T \mathbf{B}$$

Check your Understanding SOLUTION

Suppose we have

$$\mathbf{A}_{4 \times 6} \quad \mathbf{B}_{5 \times 5} \quad \mathbf{M}_{5 \times 4} \quad \mathbf{P}_{6 \times 5}$$

Circle the matrix products that are possible to compute and write the dimension of the result.

\mathbf{AM}

\mathbf{MA}

5×6

\mathbf{BM}

5×4

\mathbf{MB}

\mathbf{PA}

\mathbf{PM}

6×4

\mathbf{AP}

4×5

$\mathbf{A}^T \mathbf{P}$

$\mathbf{M}^T \mathbf{B}$

4×5

Check Your Understanding

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} -2 & 1 & -1 & 2 & -2 \\ 1 & -2 & 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

Determine the following matrix products, if possible.

a. \mathbf{AC}

b. \mathbf{AM}

c. \mathbf{AC}^T

Check your Understanding SOLUTION

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} -2 & 1 & -1 & 2 & -2 \\ 1 & -2 & 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

Determine the following matrix products, if possible.

a. $\mathbf{AC} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$

b. $\mathbf{AM} = \begin{pmatrix} -1 & -1 & -1 & 1 & 0 \\ 1 & -2 & 0 & -1 & 2 \end{pmatrix}$

c. $\mathbf{AC}^T = \text{not possible}$

NOT Commutative

- ▶ Very important to remember that

Matrix multiplication is **NOT** commutative!

- ▶ As we see in previous exercise, common to be able to compute product AB when the reverse product, BA , is not even defined.
- ▶ Even when both products are possible, almost *never* the case that $AB = BA$.

Diagonal Scaling

Multiplication by a diagonal matrix

Multiplication by a diagonal matrix

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

The net effect is that the rows of \mathbf{A} are scaled by the corresponding diagonal element of \mathbf{D}

Multiplication by a diagonal matrix

Rather than computing DA , what if we instead put the diagonal matrix on the right hand side and compute AD ?

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\mathbf{AD} =$$

(Exercise)

Matrix-Matrix Multiplication

As a Collection of Linear Combinations (L-C View)

Matrix-Matrix Multiplication (L-C View)

- ▶ Just a collection of matrix-vector products (linear combinations) with different coefficients.
- ▶ Each linear combination involves the same set of vectors (the green columns) with different coefficients (the purple columns).

Matrix-Matrix Multiplication (L-C View)

The diagram illustrates the L-C (Loop-Column) view of matrix multiplication. It shows the multiplication of two matrices, resulting in a third matrix.

The first matrix is a 4x3 matrix, represented by a grid of 12 green squares. The second matrix is a 3x4 matrix, represented by a grid of 12 purple squares. The result is a 4x4 matrix, represented by a grid of 16 purple squares.

The equation is shown as:

$$\begin{pmatrix} \text{Green Grid} \end{pmatrix} \begin{pmatrix} \text{Purple Grid} \end{pmatrix} = \begin{pmatrix} \text{Purple Grid} \end{pmatrix}$$

Matrix-Matrix Multiplication (L-C View)

$$\begin{pmatrix} \begin{array}{|c|c|c|} \hline \text{light purple} & \text{medium purple} & \text{dark purple} \\ \hline \text{light purple} & \text{medium purple} & \text{dark purple} \\ \hline \text{light purple} & \text{medium purple} & \text{dark purple} \\ \hline \end{array} \end{pmatrix} = \begin{pmatrix} \begin{array}{|c|c|c|} \hline \text{light purple} & \text{medium purple} & \text{dark purple} \\ \hline \text{light purple} & \text{medium purple} & \text{dark purple} \\ \hline \text{light purple} & \text{medium purple} & \text{dark purple} \\ \hline \text{light purple} & \text{medium purple} & \text{dark purple} \\ \hline \end{array} \end{pmatrix}$$

$$\begin{pmatrix} \begin{array}{|c|c|c|} \hline \text{green} & \text{green} & \text{green} \\ \hline \text{green} & \text{green} & \text{green} \\ \hline \text{green} & \text{green} & \text{green} \\ \hline \text{green} & \text{green} & \text{green} \\ \hline \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{|c|} \hline \text{light purple} \\ \hline \text{light purple} \\ \hline \text{light purple} \\ \hline \text{light purple} \\ \hline \end{array} \end{pmatrix} = \begin{pmatrix} \begin{array}{|c|} \hline \text{light purple} \\ \hline \text{light purple} \\ \hline \text{light purple} \\ \hline \text{light purple} \\ \hline \end{array} \end{pmatrix}$$

Matrix-Matrix Multiplication (L-C View)

$$\begin{array}{c}
 \left(\begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} \right) + \left(\begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} \right) + \left(\begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} \right) \left(\begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \right) = \left(\begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{array} \right) \\
 \left. \sum_{i=1}^n \left(\begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} \right) \mathbf{a}_{i1} \right\} = \left(\begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} \right)
 \end{array}$$

The diagram illustrates the L-C (Loop Column) view of matrix-matrix multiplication. It shows the accumulation of the first column of the product matrix. The left side represents the summation of the first column of the product matrix, which is the sum of the first column of the input matrix multiplied by each element of the first row of the input matrix. The right side shows the resulting first column of the product matrix, which is a column vector of four light purple squares.

Matrix-Matrix Multiplication (L-C View)

$$\begin{pmatrix} \begin{array}{cc} \text{light purple} & \text{medium purple} \\ \text{light purple} & \text{medium purple} \\ \text{light purple} & \text{medium purple} \end{array} & \begin{array}{c} \text{dark purple} \\ \text{dark purple} \\ \text{dark purple} \end{array} \end{pmatrix} = \begin{pmatrix} \begin{array}{cc} \text{light purple} & \text{medium purple} \\ \text{light purple} & \text{medium purple} \\ \text{light purple} & \text{medium purple} \\ \text{light purple} & \text{medium purple} \end{array} & \begin{array}{c} \text{dark purple} \\ \text{dark purple} \\ \text{dark purple} \\ \text{dark purple} \end{array} \end{pmatrix}$$

$$\begin{pmatrix} \text{medium purple} & \begin{array}{c} \text{green} \\ \text{green} \\ \text{green} \\ \text{green} \end{array} \end{pmatrix} + \begin{pmatrix} \text{medium purple} & \begin{array}{c} \text{green} \\ \text{green} \\ \text{green} \\ \text{green} \end{array} \end{pmatrix} + \begin{pmatrix} \text{medium purple} & \begin{array}{c} \text{green} \\ \text{green} \\ \text{green} \\ \text{green} \end{array} \end{pmatrix} = \begin{pmatrix} \text{purple} \\ \text{purple} \\ \text{purple} \\ \text{purple} \end{pmatrix}$$

Matrix-Matrix Multiplication (L-C View)

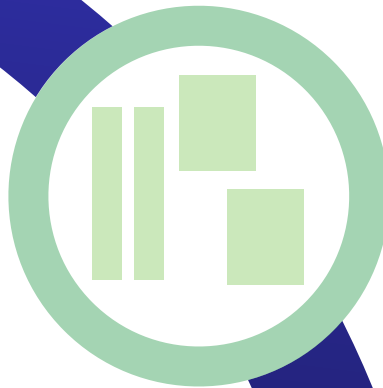
$$\begin{pmatrix} \text{light purple} & \text{medium purple} & \text{dark purple} \\ \text{light purple} & \text{medium purple} & \text{dark purple} \\ \text{light purple} & \text{medium purple} & \text{dark purple} \end{pmatrix} = \begin{pmatrix} \text{light purple} & \text{medium purple} & \text{dark purple} \\ \text{light purple} & \text{medium purple} & \text{dark purple} \\ \text{light purple} & \text{medium purple} & \text{dark purple} \\ \text{light purple} & \text{medium purple} & \text{dark purple} \end{pmatrix}$$

$$\begin{pmatrix} \text{dark purple} \\ \text{light green} \\ \text{light green} \\ \text{light green} \\ \text{light green} \end{pmatrix} + \begin{pmatrix} \text{dark purple} \\ \text{light green} \\ \text{light green} \\ \text{light green} \\ \text{light green} \end{pmatrix} + \begin{pmatrix} \text{dark purple} \\ \text{light green} \\ \text{light green} \\ \text{light green} \\ \text{light green} \end{pmatrix} = \begin{pmatrix} \text{dark purple} \\ \text{dark purple} \\ \text{dark purple} \\ \text{dark purple} \\ \text{dark purple} \end{pmatrix}$$

Matrix-Matrix Multiplication (L-C View)

- ▶ Just a collection of matrix-vector products (linear combinations) with different coefficients.
- ▶ Each linear combination involves the same set of vectors (the green columns) with different coefficients (the purple columns).
- ▶ This has important implications!

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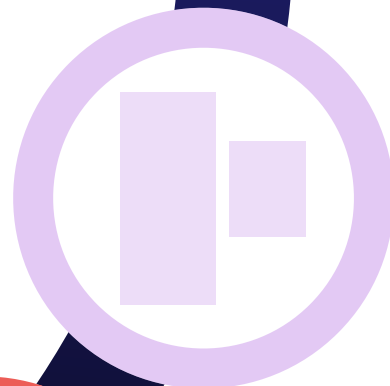
Element-wise Operations

Linear Combinations of Matrices and Vectors.



Vector Multiplication

Inner products and Matrix-Vector Multiplication



Matrix Multiplication

Inner product and linear combination viewpoint

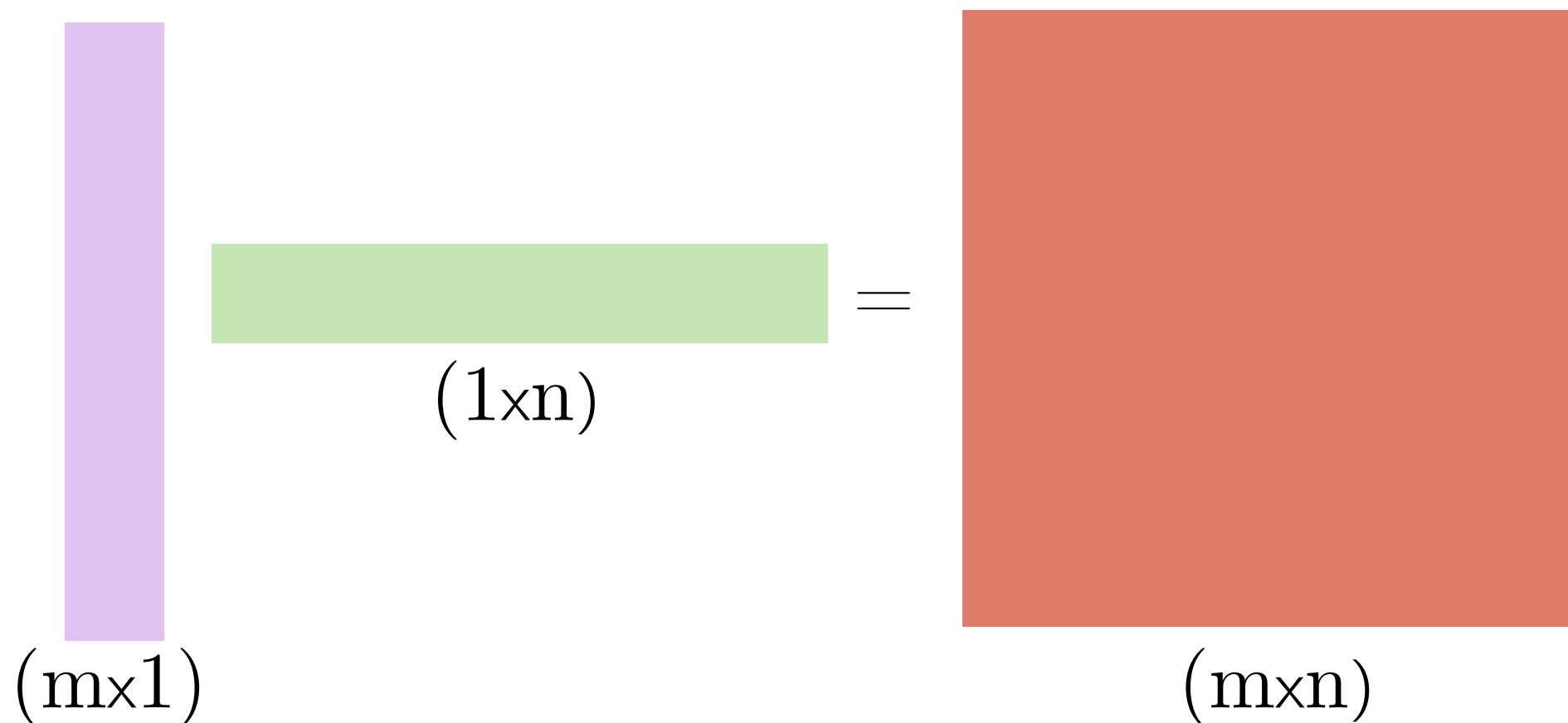


Vector Multiplication

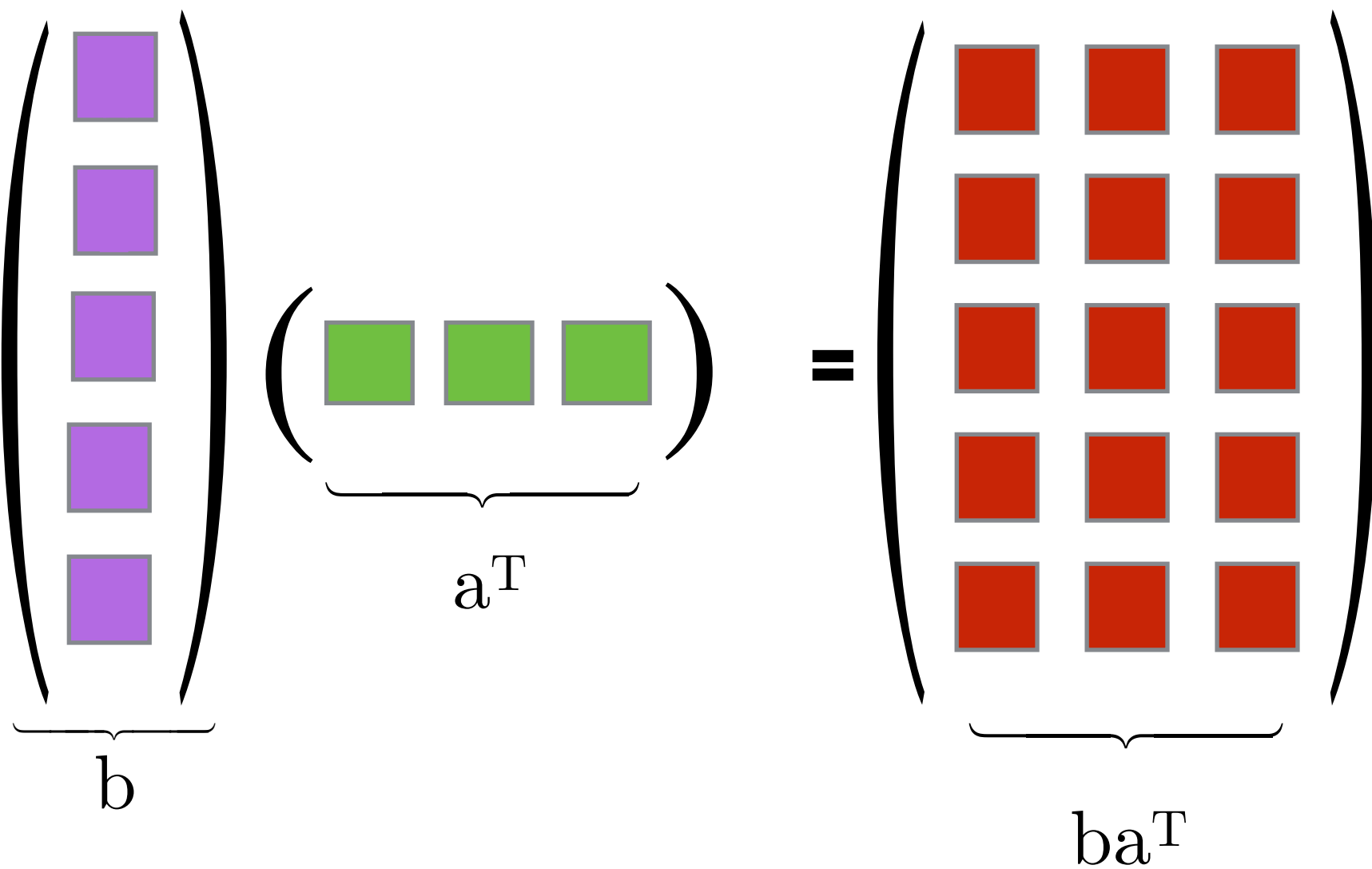
The Outer Product

Vector Outer Product

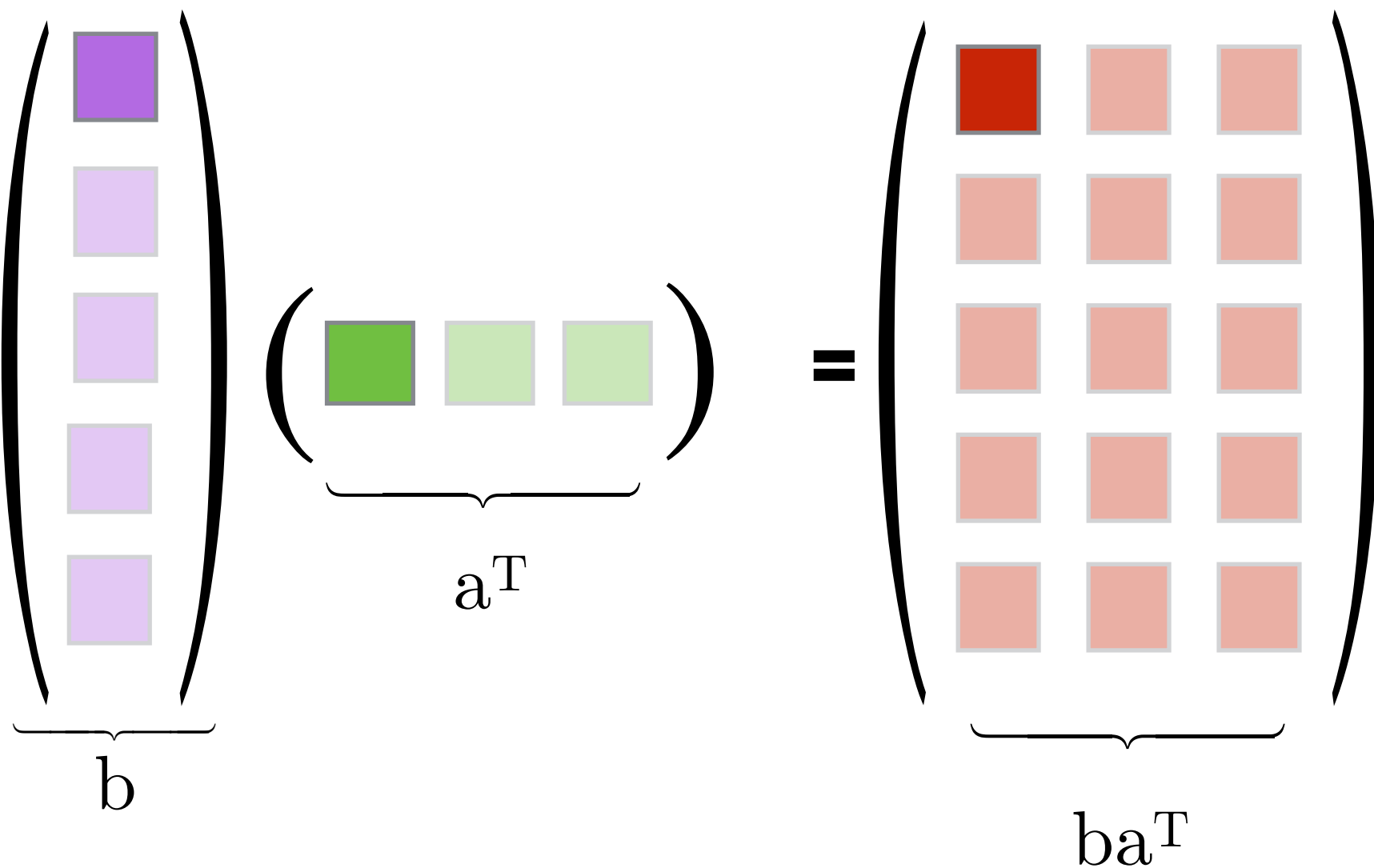
- ▶ The vector outer product is the multiplication of a column vector times a row vector.
- ▶ For any column/row this product is possible
- ▶ The result of this product is a *matrix*!



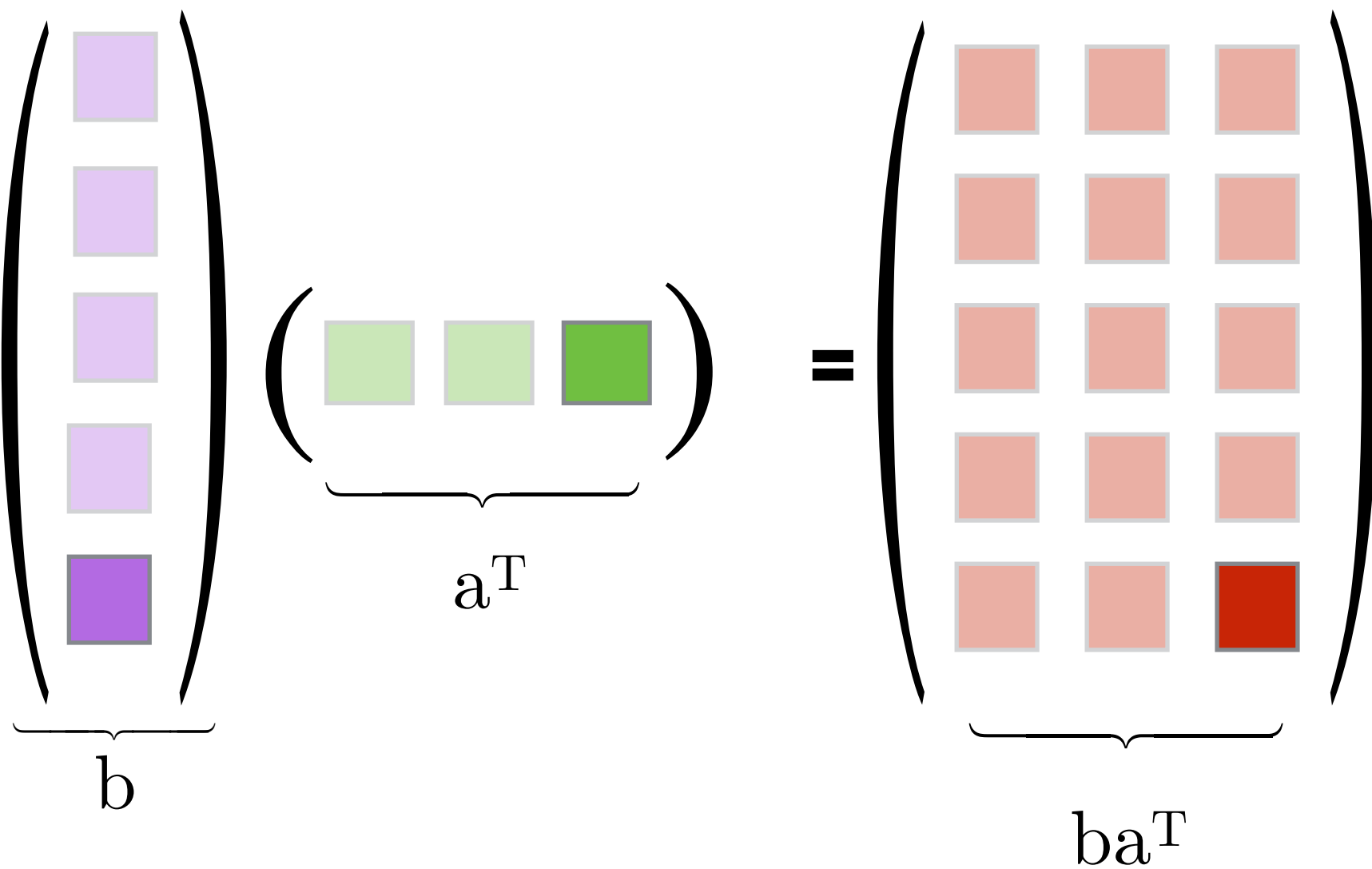
Outer Product (column \times row)



Outer Product (column \times row)



Outer Product (column \times row)



Example: Outer Product

Let $\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$. Then,

$$\mathbf{xy}^T = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \begin{pmatrix} 1 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 15 & 9 \\ 4 & 20 & 12 \\ -2 & -10 & -6 \end{pmatrix}$$

Outer Product has rank 1

From the previous example, you can see that the rows of an outer product are necessarily multiples of each other.

The diagram illustrates the outer product of a column vector b and a row vector a^T . On the left, a column vector b is represented by a vertical stack of five purple squares, enclosed in large parentheses, with a brace underneath labeled b . To its right is a row vector a^T , represented by a horizontal stack of three green squares, enclosed in large parentheses, with a brace underneath labeled a^T . An equals sign follows, leading to the resulting matrix ba^T . This matrix is a 5x3 grid of squares, enclosed in large parentheses, with a brace underneath labeled ba^T . The squares in the first and fifth rows are red, while the squares in the second, third, and fourth rows are light red, visually demonstrating that all rows are scalar multiples of the first row.

Matrix-Matrix Multiplication

As a Sum of Outer Products (O-P View)

Matrix-Matrix Multiplication (O-P View)

We can write the product \mathbf{AB} as a *sum of outer products* of columns of $\mathbf{A}_{(m \times n)}$ and rows of $\mathbf{B}_{(n \times p)}$

$$\mathbf{AB} = \sum_{i=1}^n \mathbf{A}_{\star i} \mathbf{B}_{i \star}$$

This view decomposes the product \mathbf{AB} into the sum of n matrices, each of which has rank 1 (discussed later).

Challenge Puzzle

- Suppose we have 1,000 individuals that have been divided into 5 different groups each year for 20 years.
- We need to make a 1000x1000 matrix C where
$$C_{ij} = \# \text{ times person } i \text{ grouped with person } j$$
- The data currently has 1000 rows and $5 \times 20 = 100$ binary columns indicating whether each individual was a member of each group (yLgK: yearLgroupK):
(y1g1, y1g2, y1g3, y1g4, y1g5, y2g1, ... y20g5)
- Can we use what we've just learned to help us here?

Special Cases of Matrix Multiplication

The Identity and the Inverse

The Identity Matrix

- ▶ The identity matrix, 'I', is to matrices what the number '1' is to scalars.

- ▶ It is the multiplicative identity.

$$\mathbf{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- ▶ For any matrix (or vector) A, multiplying A by the (appropriately sized) identity matrix on either side does not change A:

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

The Matrix Inverse

- ▶ For certain square matrices, \mathbf{A} , an inverse matrix written \mathbf{A}^{-1} , exists such that:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

- ▶ **A MATRIX MUST BE SQUARE TO HAVE AN INVERSE.**
 - ▶ **NOT ALL SQUARE MATRICES HAVE AN INVERSE**
 - ▶ Only full-rank, square matrices are invertible. (more on this later)
- ▶ For now, understand that the inverse matrix serves like the multiplicative inverse in scalar algebra:

$$(2)(2^{-1}) = (2)\left(\frac{1}{2}\right) = 1$$

- ▶ Multiplying a matrix by its inverse (if it exists) yields the multiplicative identity, \mathbf{I}

The Matrix Inverse

- ▶ A *square* matrix which has an inverse is equivalently called:
 - ▶ Non-singular
 - ▶ Invertible
 - ▶ Full Rank

Don't Cancel That!!

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

Cancellation implies inversion

Canceling numbers in scalar algebra:

$$2x = 2y$$

$$\cancel{2}x = \cancel{2}y$$

$$\frac{1}{\cancel{2}}\cancel{2}x = \frac{1}{\cancel{2}}\cancel{2}y$$

$$1x = 1y$$

Canceling matrices in linear algebra:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{I}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Inverses can help
solve an equation...

WHEN THEY EXIST!