Matrix Arithmetic

(Multidimensional Math)

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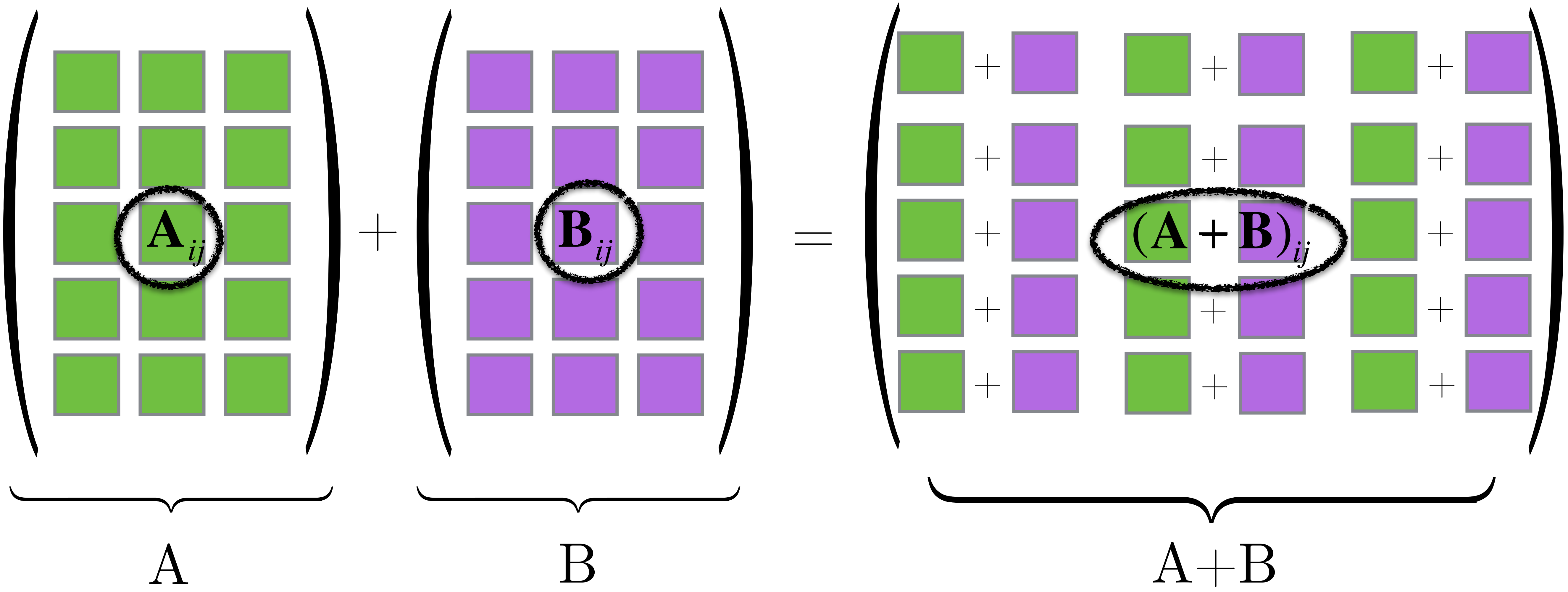
Matrix Addition/Subtraction

- Two matrices/vectors can be added/subtracted if and only if they have the same size
- Then simply add/subtract corresponding elements

Matrix Addition/Subtraction



$$\boxed{(\mathbf{A}+\mathbf{B})_{ij} = \mathbf{A}_{ij} + \mathbf{B}_{ij}}$$

(Element-wise)

Example: Matrix Addition/Subtraction

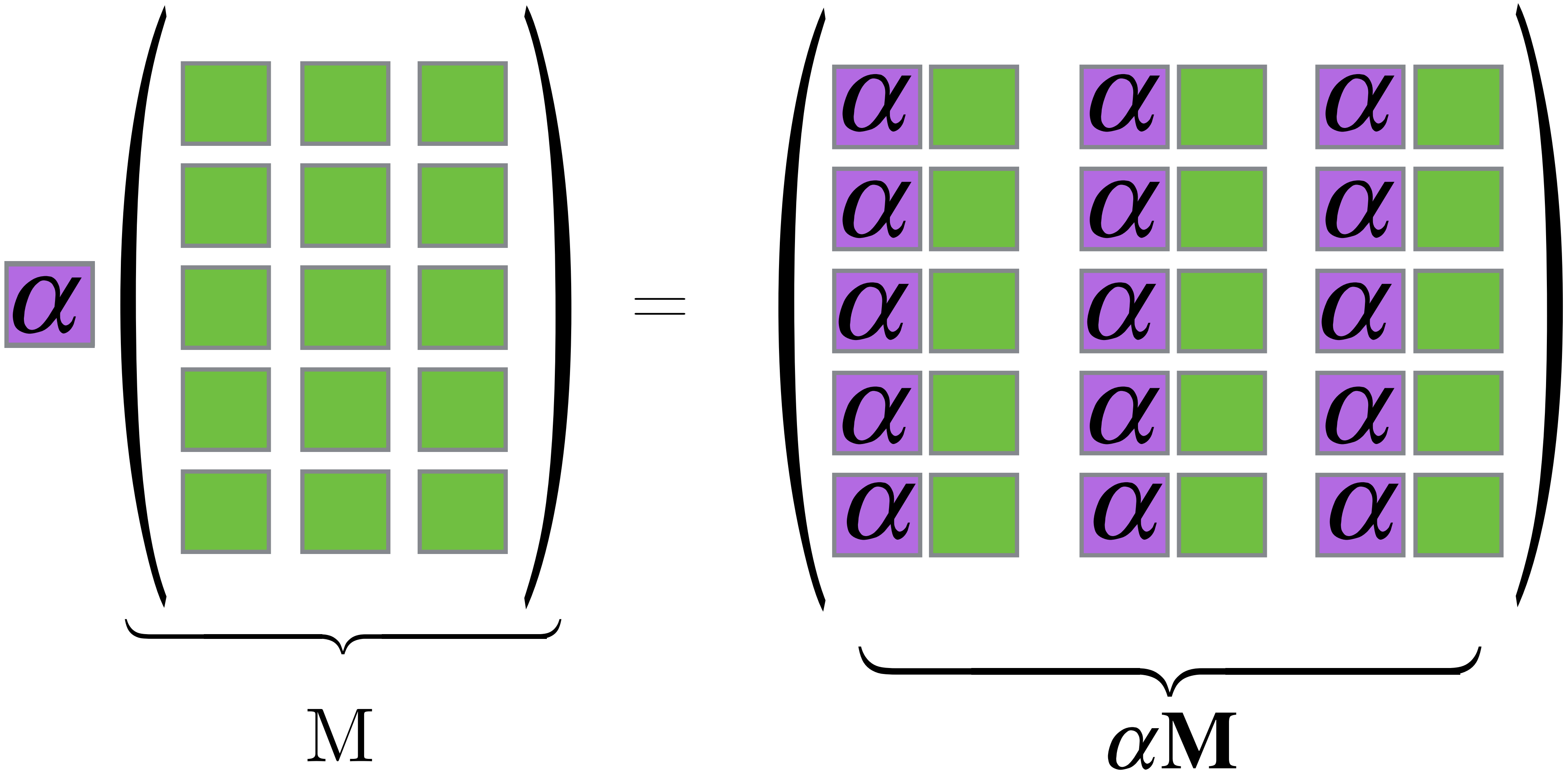
a) Compute $\mathbf{A} + \mathbf{B}$, if possible:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 4 & 5 & 6 \\ -1 & 0 & 4 \\ 3 & 4 & 3 \end{pmatrix}$$

b) Compute $\mathbf{A} - \mathbf{H}$, if possible:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \qquad \mathbf{H} = \begin{pmatrix} 6 & 5 & 10 \\ 0.1 & 0.5 & 0.9 \end{pmatrix}$$

Scalar Multiplication



$$\boxed{(\alpha\mathbf{M})_{ij} = \alpha\mathbf{M}_{ij}}$$

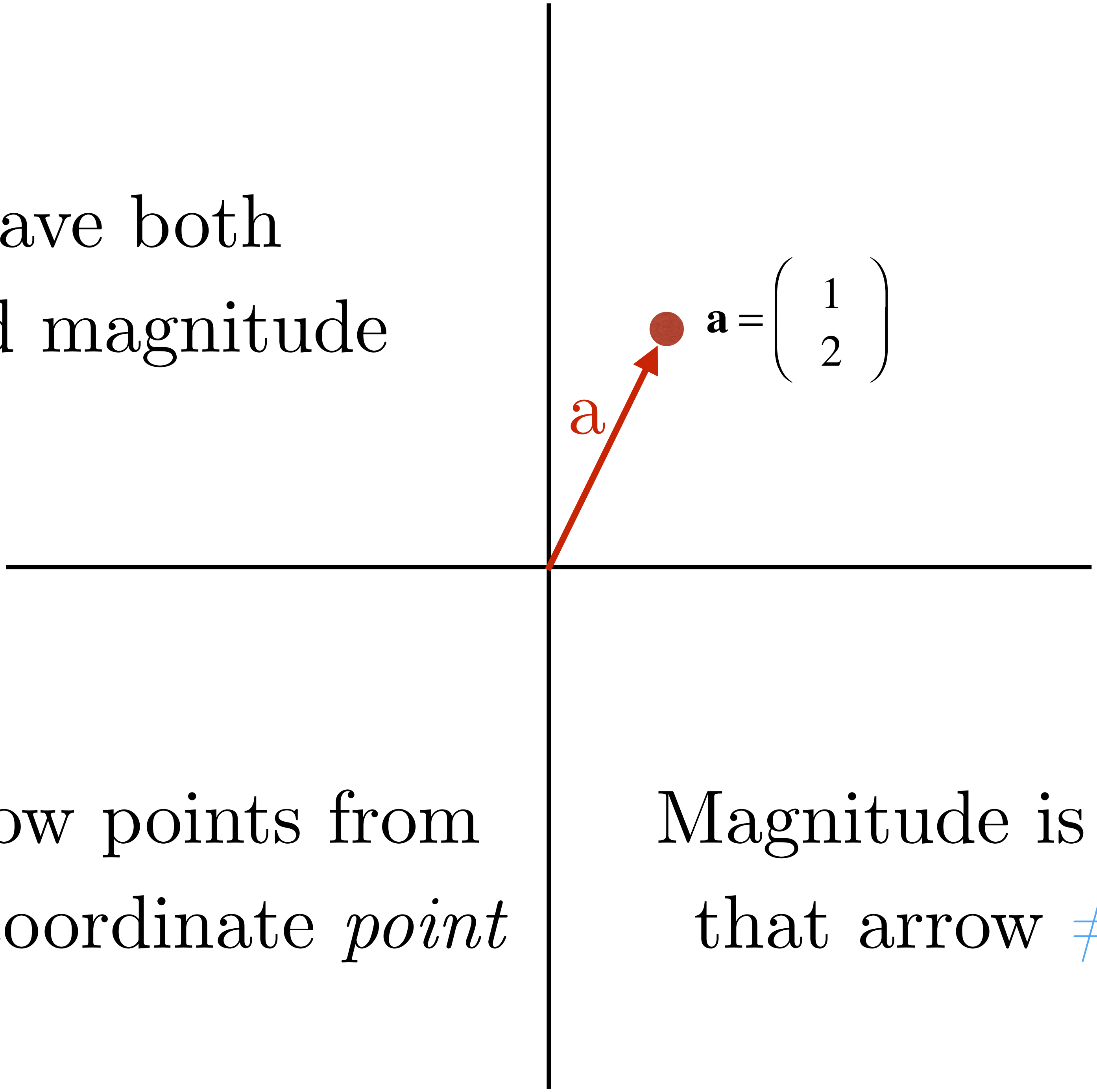
(Element-wise)

Geometric Look

Vector addition and scalar multiplication

Points <—> Vectors

Vectors have both direction and magnitude

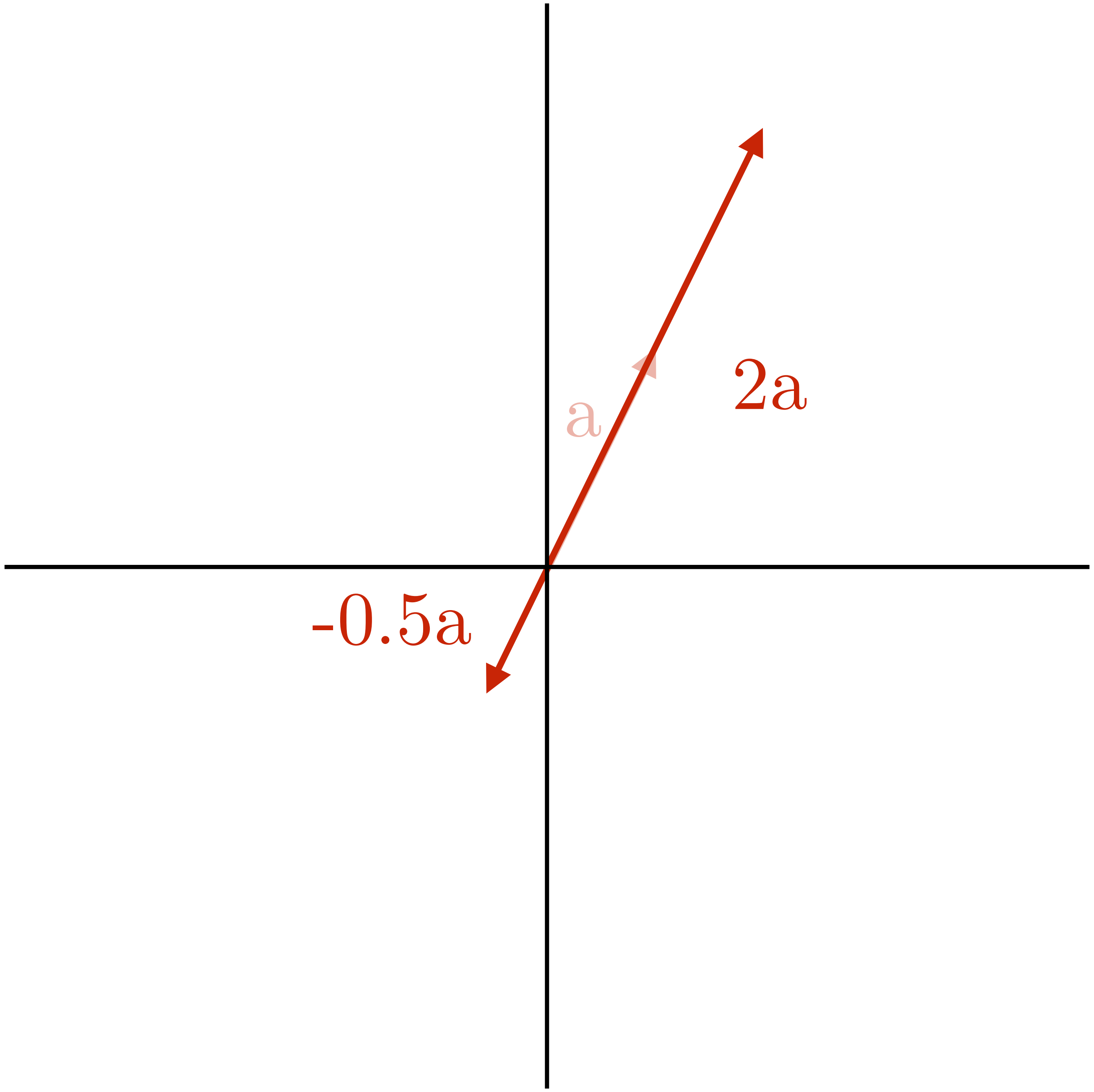


$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

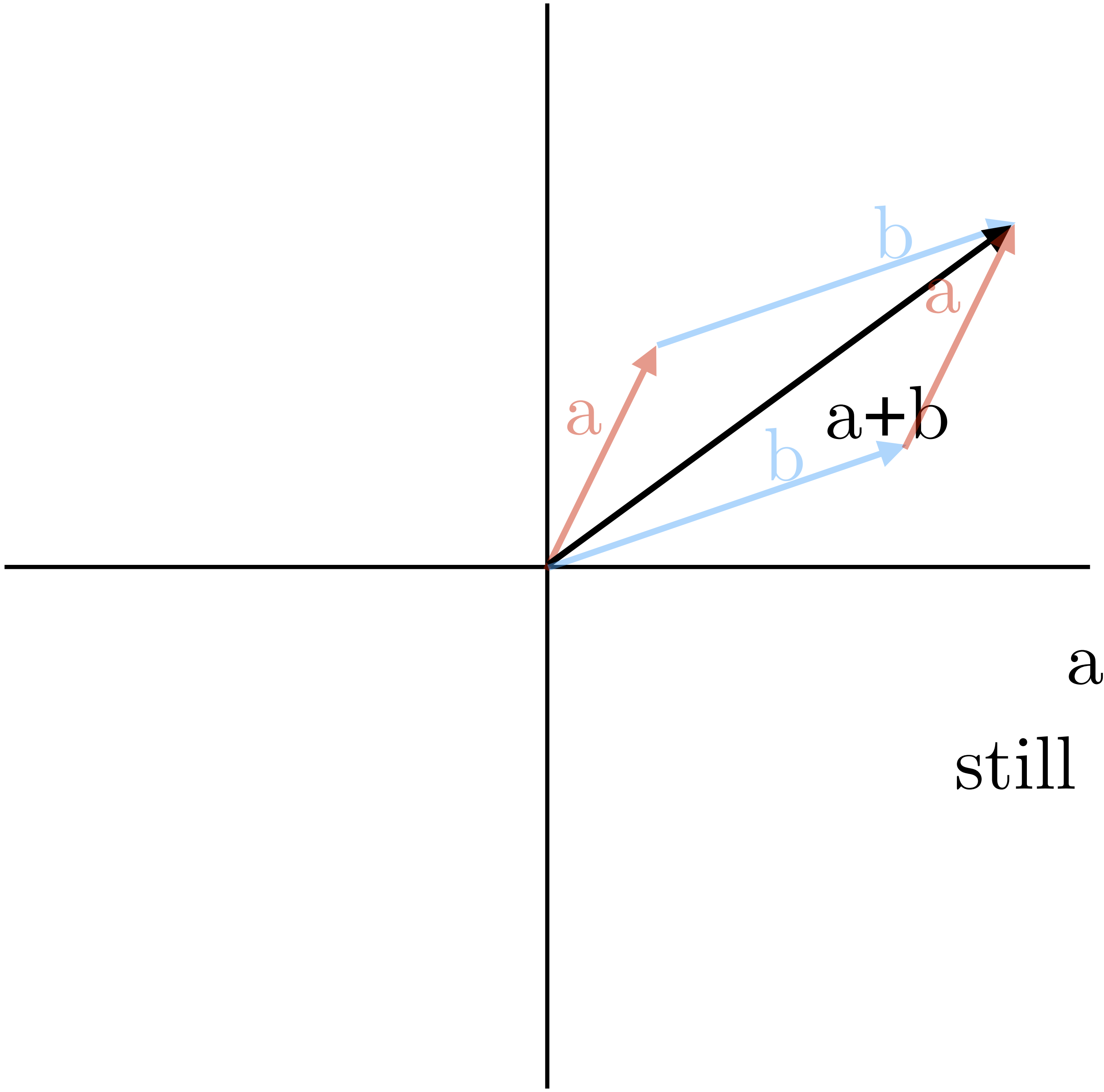
Direction arrow points from origin to the coordinate *point*

Magnitude is the length of that arrow #pythagoras

Scalar Multiplication (Geometrically)

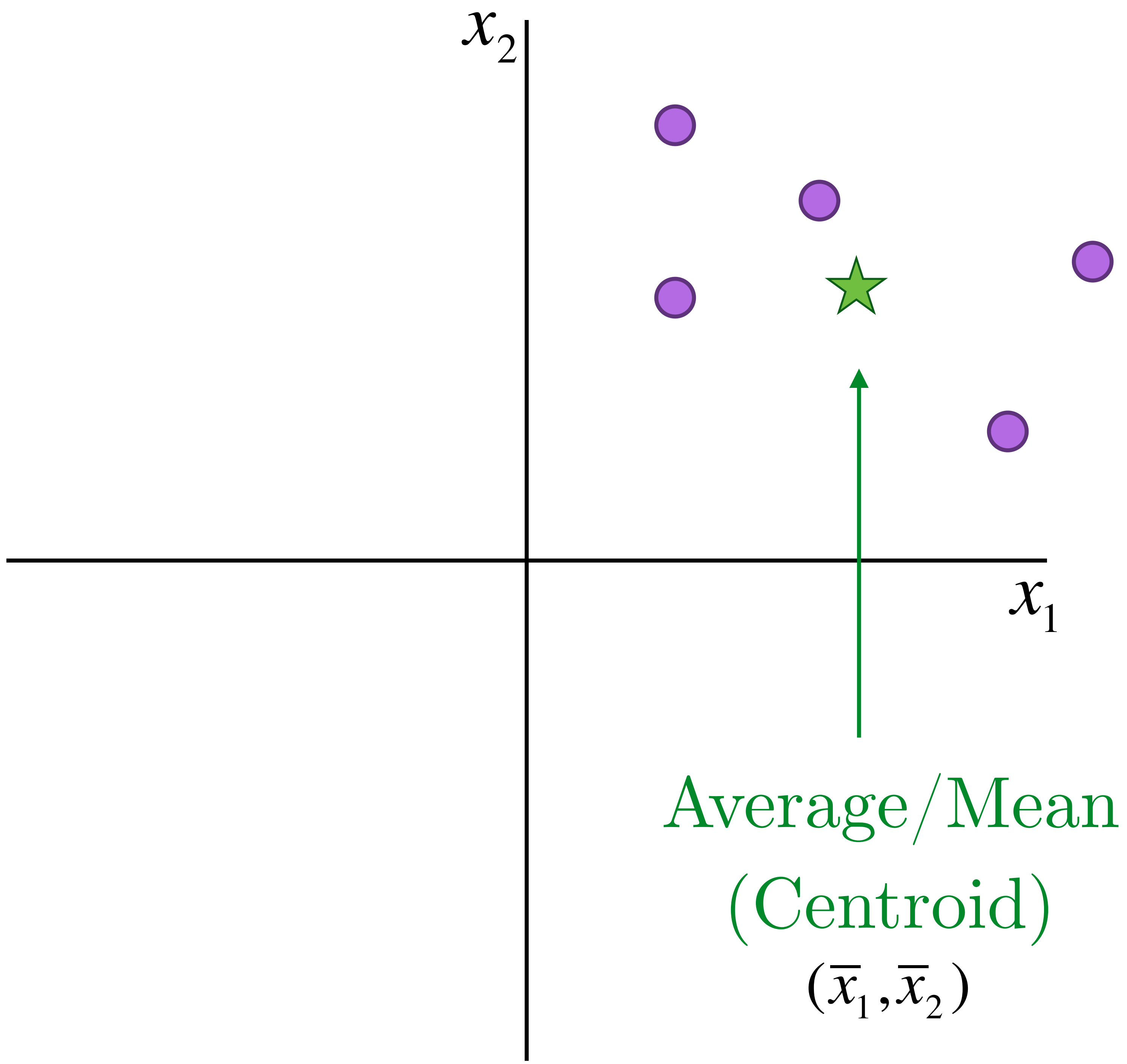


Vector Addition (Geometrically)



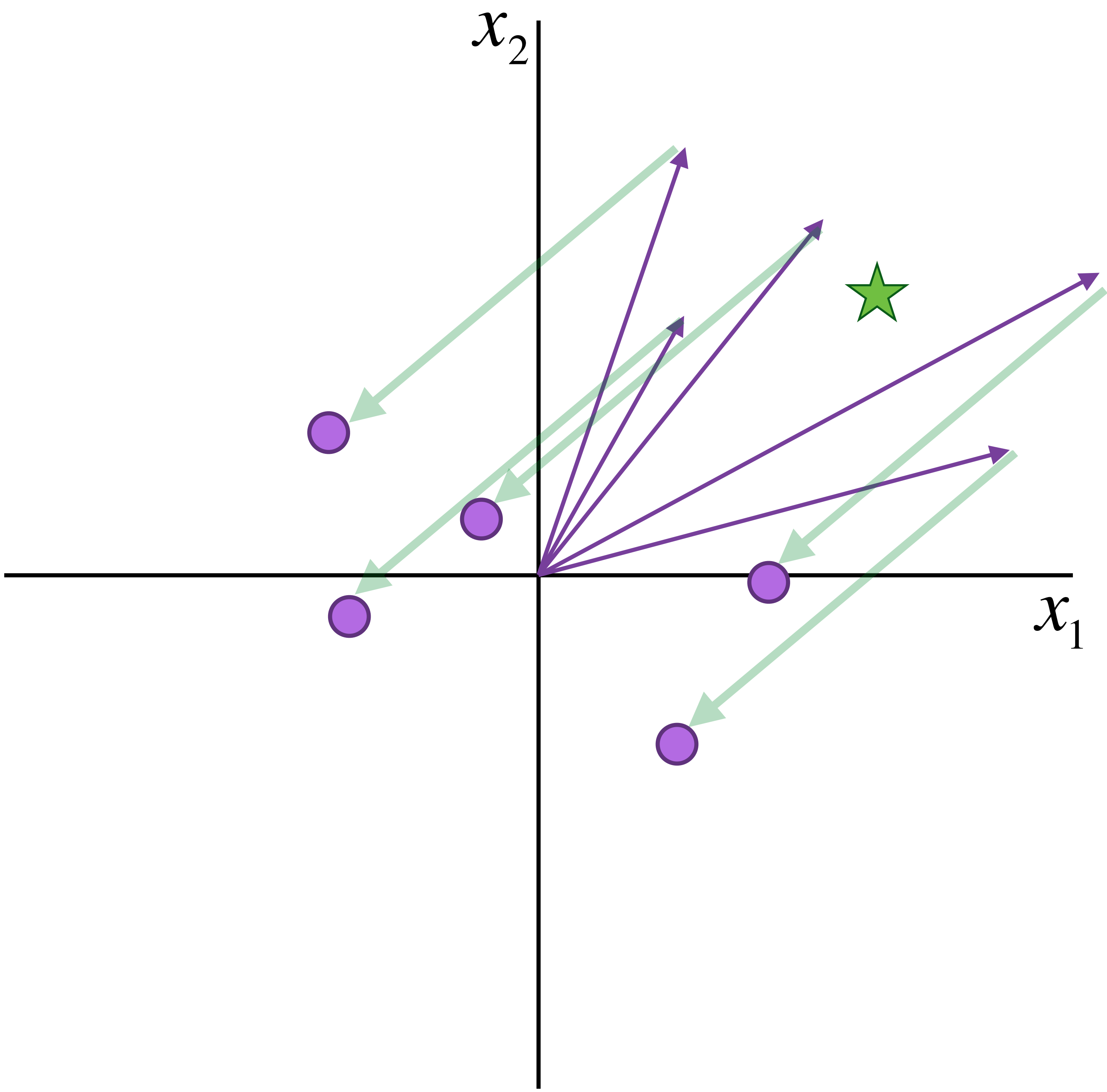
addition is
still commutative

Example: Centering the data

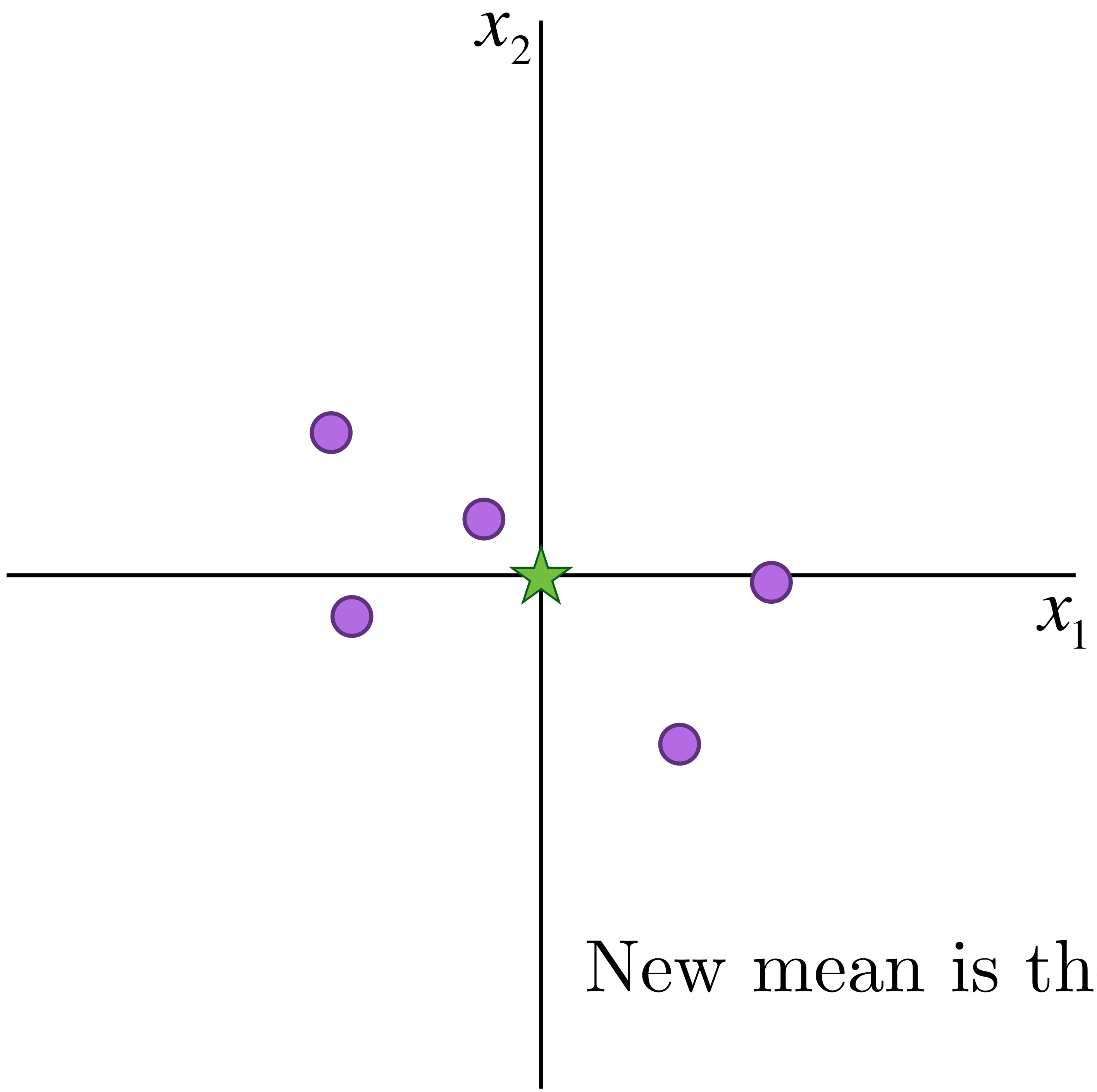


Average/Mean
(Centroid)
$(\bar{x}_1, \bar{x}_2)$

Example: Centering the data



Example: Centering the data

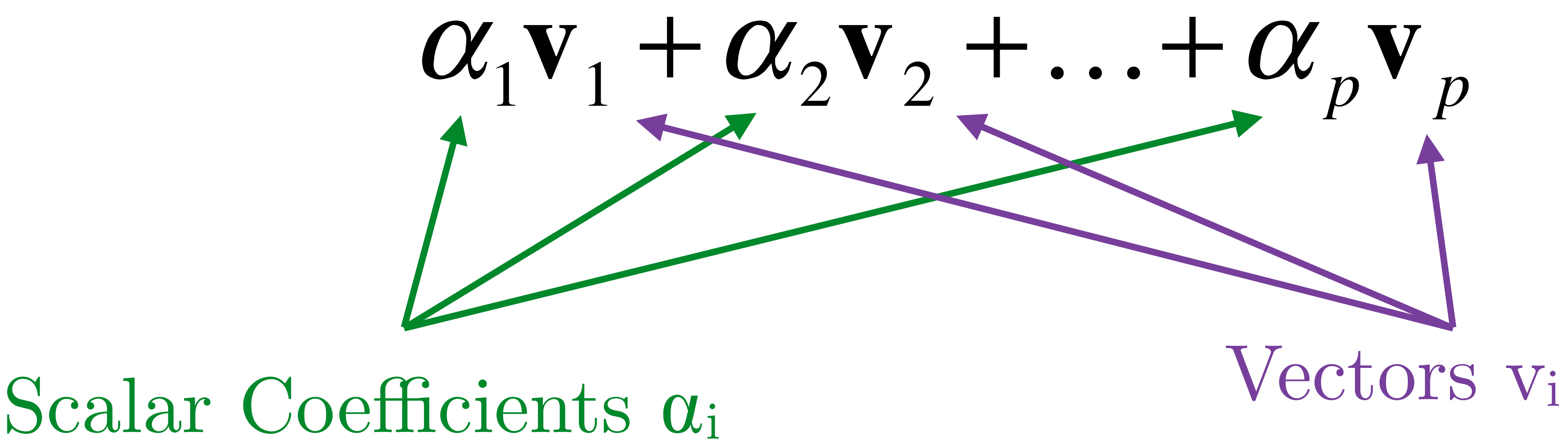


New mean is the origin (0,0)

Linear Combinations

Linear Combinations

A linear combination of vectors is a just weighted sum:



$$\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\ldots+\alpha_p\mathbf{v}_p$$

Scalar Coefficients α_i

Vectors v_i

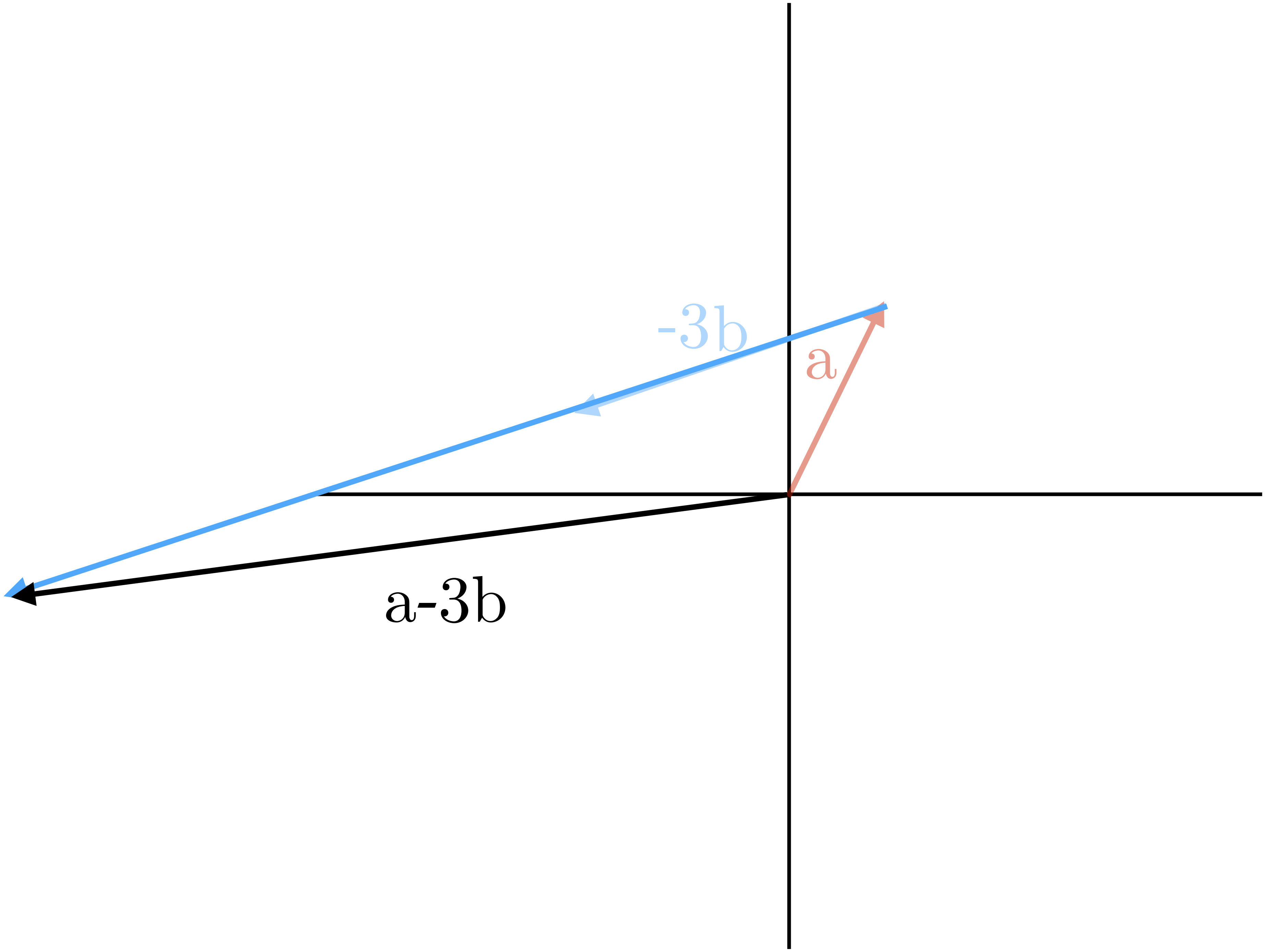
Elementary Linear Combinations

- The simplest linear combination might involve columns of the identity matrix (elementary vectors):

$$\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

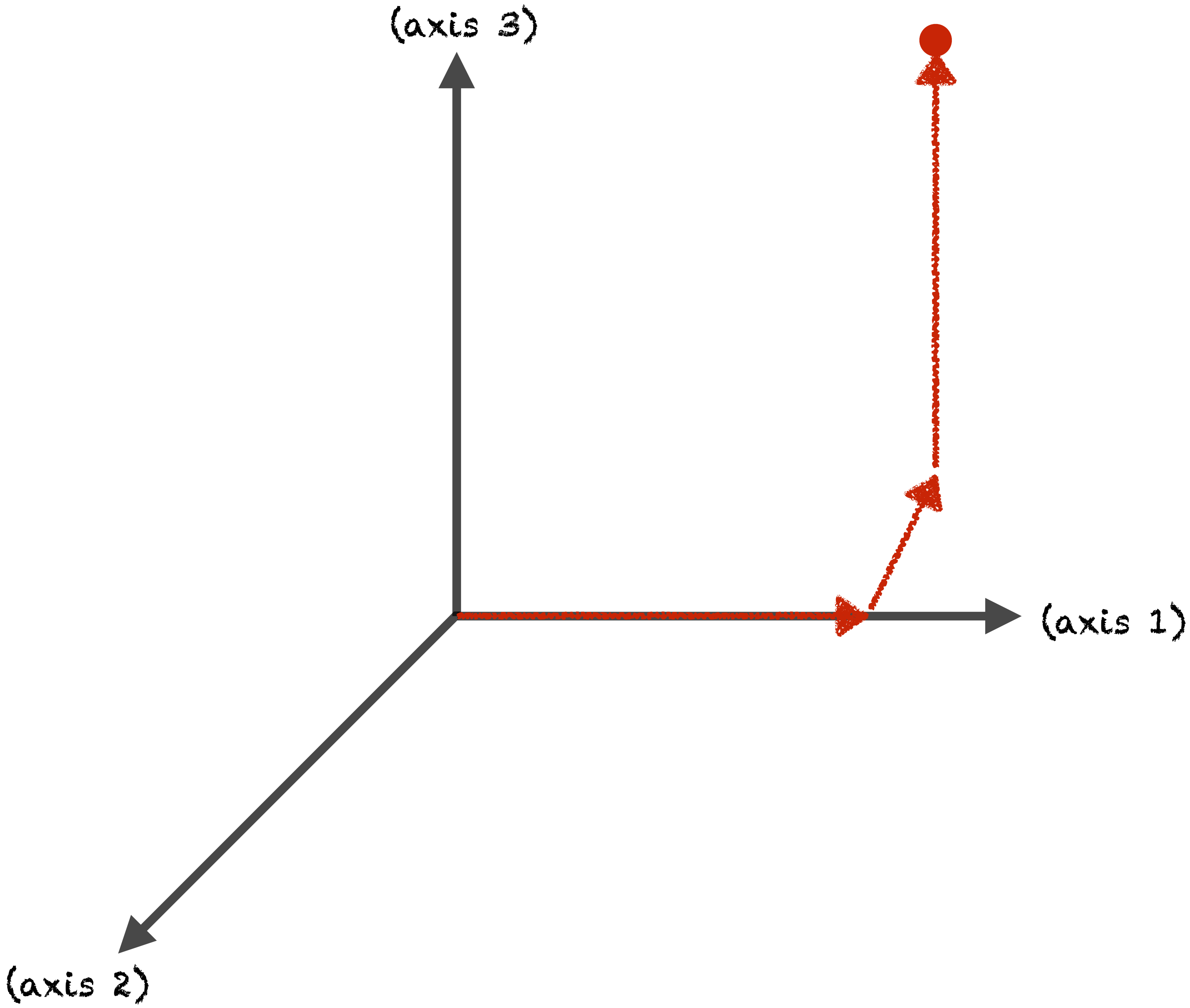
- Picture this linear combination as a "breakdown into parts" where the parts give directions along the 3 coordinate axes.

Linear Combinations (Geometrically)



Linear Combinations (Geometrically)

$$\begin{pmatrix}3\\-2\\4\end{pmatrix} = 3\begin{pmatrix}1\\0\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix}$$



Example: Linear Combination of Matrices

Write the matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ as a linear combination of the following matrices:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Solution:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

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Element-wise Operations

Linear Combinations of Matrices and Vectors.

Vector Multiplication

Inner products and Matrix-Vector Multiplication

Matrix Multiplication

Inner product and linear combination viewpoint

Vector Multiplication

The Outer Product

Notation: Column vs. Row Vectors

- Throughout this course, unless otherwise specified, all vectors are assumed to be columns.
- Simplifies notation because if x is a column vector:

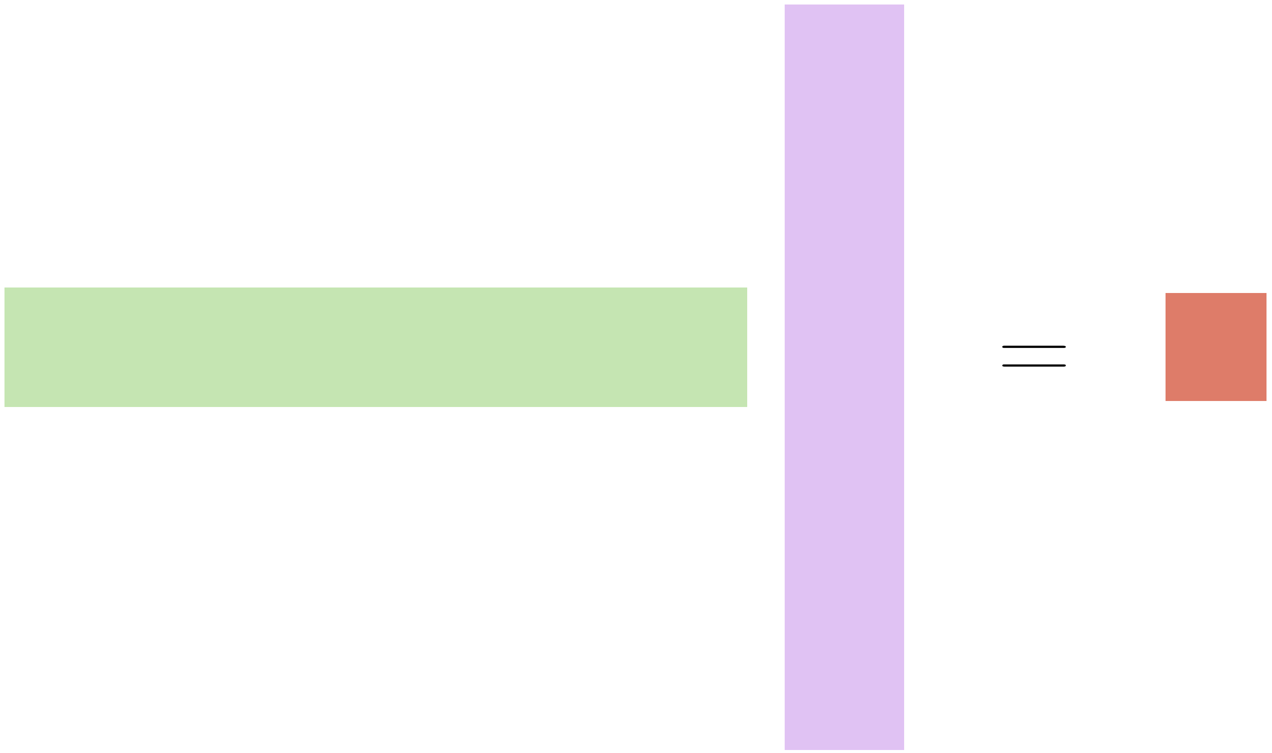
$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

then we can automatically assume that x^T is a row vector:

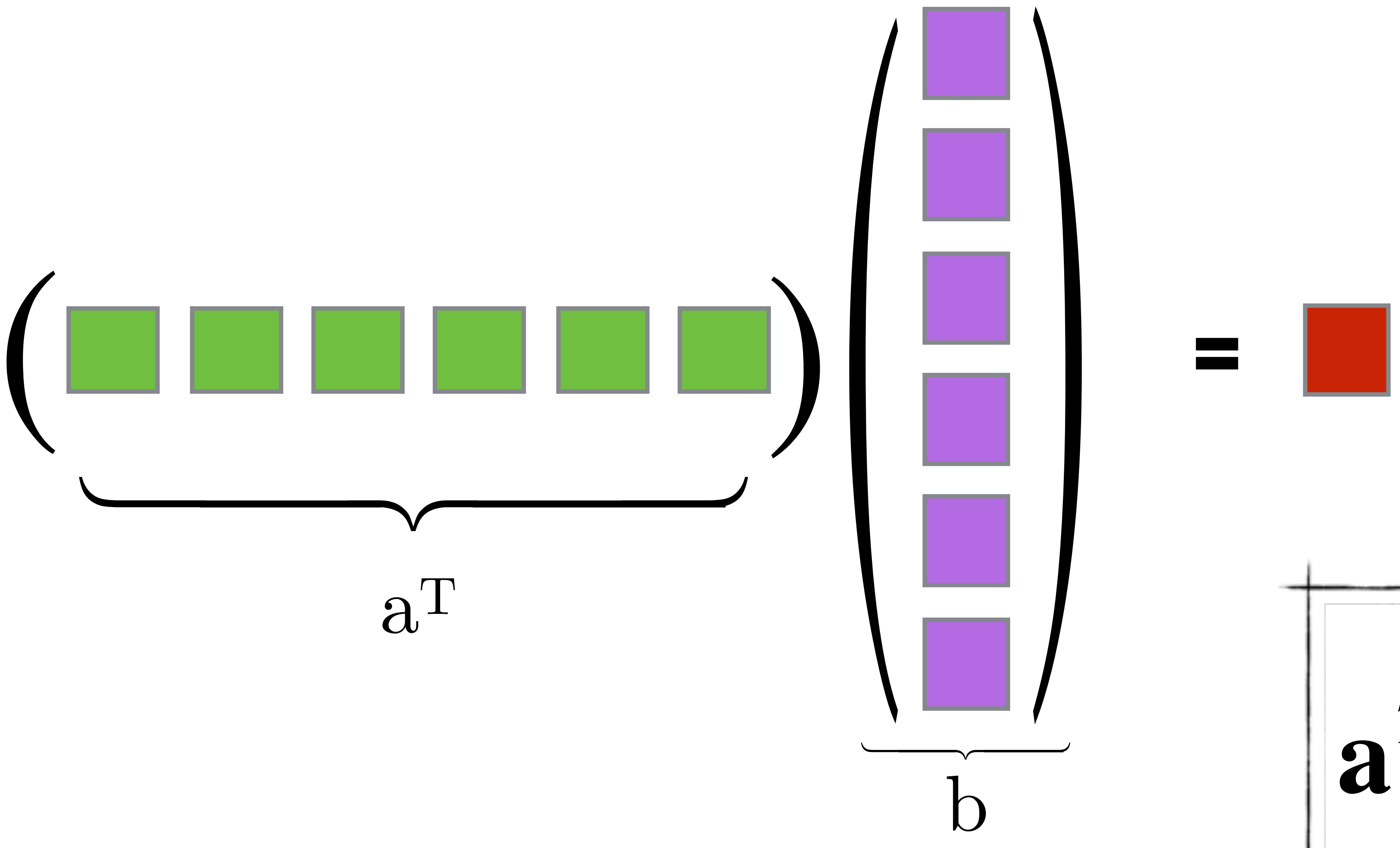
$$\mathbf{x}^T = (x_1 \; x_2 \; \ldots \; x_n)$$

Vector Inner Product

- The vector inner product is the multiplication of a row vector times a column vector.
- It is known across broader sciences as the ‘dot product’.
- The result of this product is a scalar.

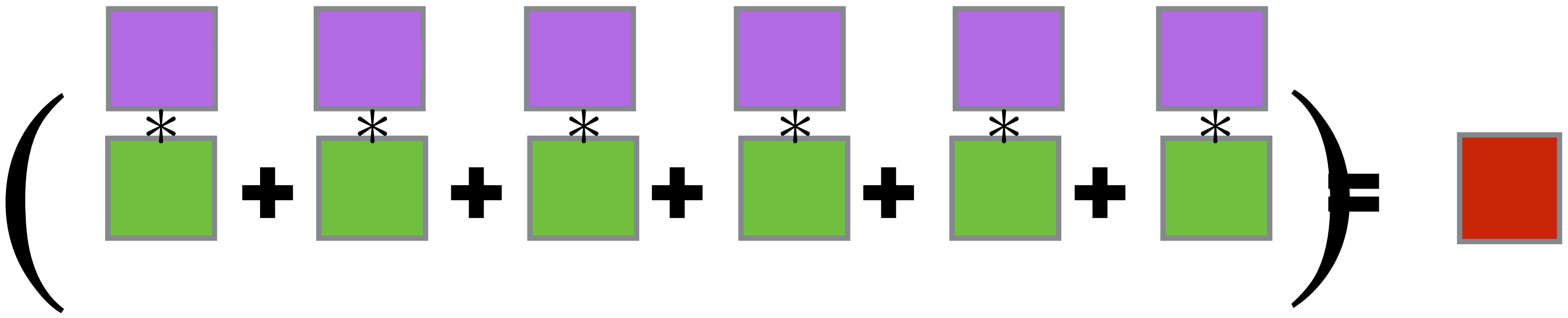


Inner Product (row x column)



$$\mathbf{a}^T\mathbf{b} = \sum_{i=1}^{n} a_i b_i$$

Inner Product (row x column)



a and **b** must have the same number of elements.

Examples: Inner Product

Let

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 4 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 3 \\ 5 \\ 1 \\ 7 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} -3 \\ -2 \\ 5 \\ 3 \\ -2 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \\ -3 \\ -2 \end{pmatrix}$$

If possible, compute the following inner products:

a. $\mathbf{x}^T\mathbf{y}$

Examples: Inner Product

Let

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 4 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 3 \\ 5 \\ 1 \\ 7 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} -3 \\ -2 \\ 5 \\ 3 \\ -2 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \\ -3 \\ -2 \end{pmatrix}$$

If possible, compute the following inner products:

b. $\mathbf{x}^T\mathbf{v}$

c. $\mathbf{v}^T\mathbf{u}$

Check your Understanding

Let

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{p} = \begin{pmatrix} 0.5 \\ 0.1 \\ 0.2 \\ 0 \\ 0.2 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 10 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{s} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

If possible, compute the following inner products:

a. $\mathbf{v}^T\mathbf{e}$

b. $\mathbf{e}^T\mathbf{v}$

c. $\mathbf{v}^T\mathbf{s}$

d. $\mathbf{p}^T\mathbf{u}$

e. $\mathbf{v}^T\mathbf{v}$

Check your Understanding SOLUTION

Let

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{p} = \begin{pmatrix} 0.5 \\ 0.1 \\ 0.2 \\ 0 \\ 0.2 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} 10 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{s} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

If possible, compute the following inner products:

a. $\mathbf{v}^T\mathbf{e} = 15$

b. $\mathbf{e}^T\mathbf{v} = 15$

c. $\mathbf{v}^T\mathbf{s} =$ *not possible*

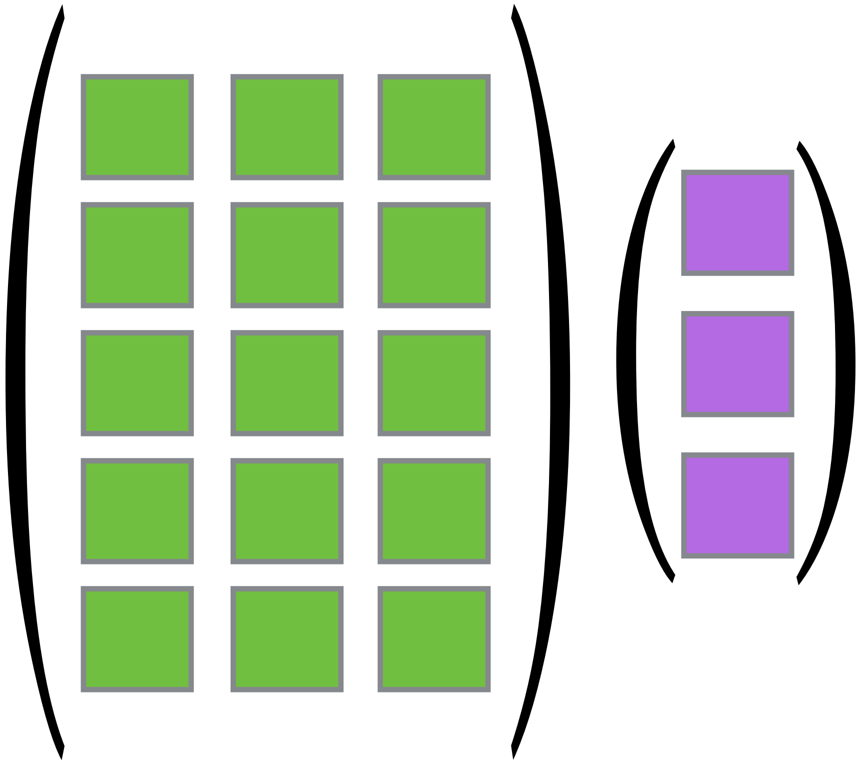
d. $\mathbf{p}^T\mathbf{u} = 6.2$

e. $\mathbf{v}^T\mathbf{v} = 55$

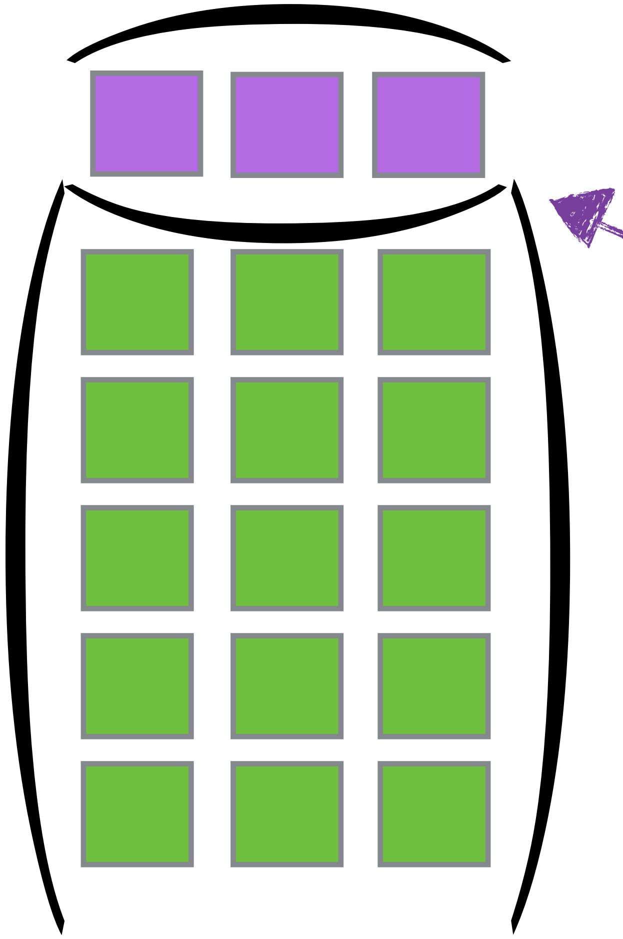
Matrix-Vector Multiplication

Inner Product View (I-P View)

Matrix-Vector Multiplication (I-P view)

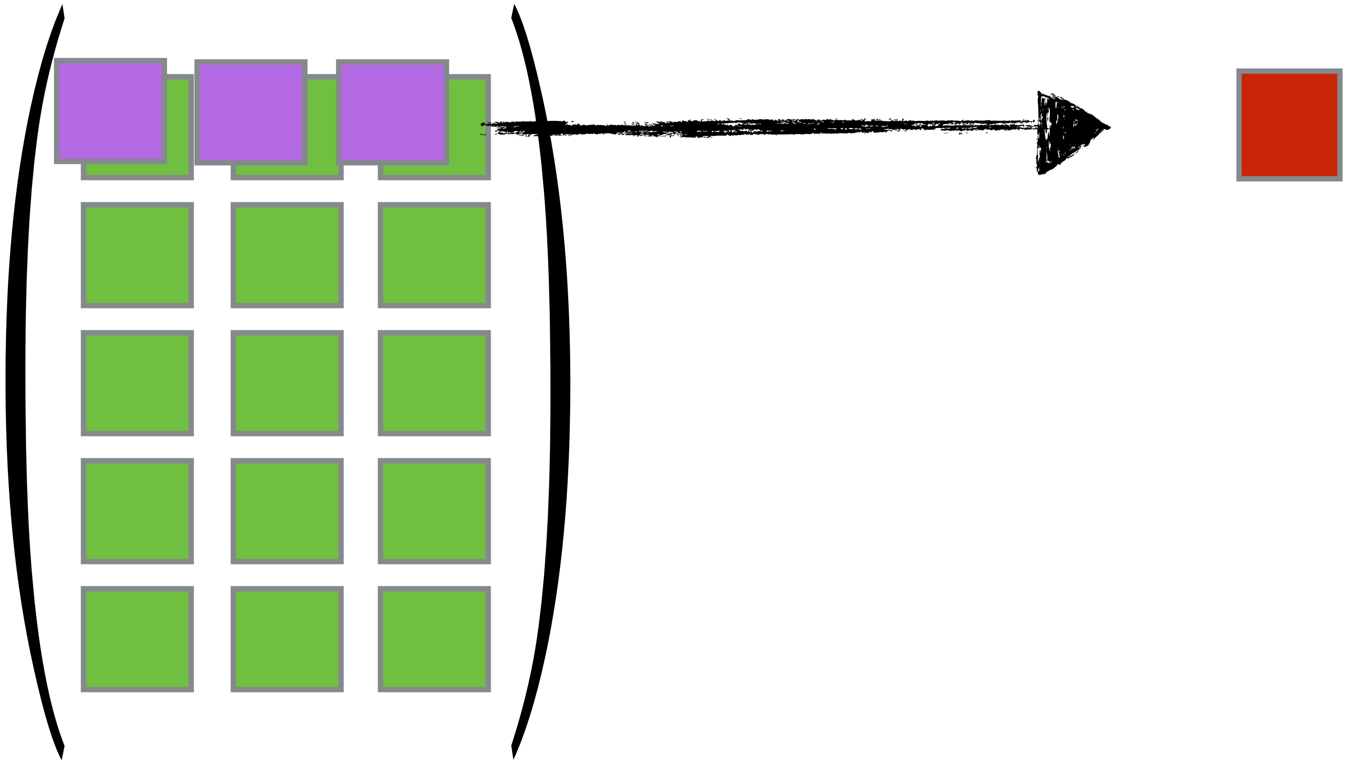


Matrix-Vector Multiplication (I-P view)

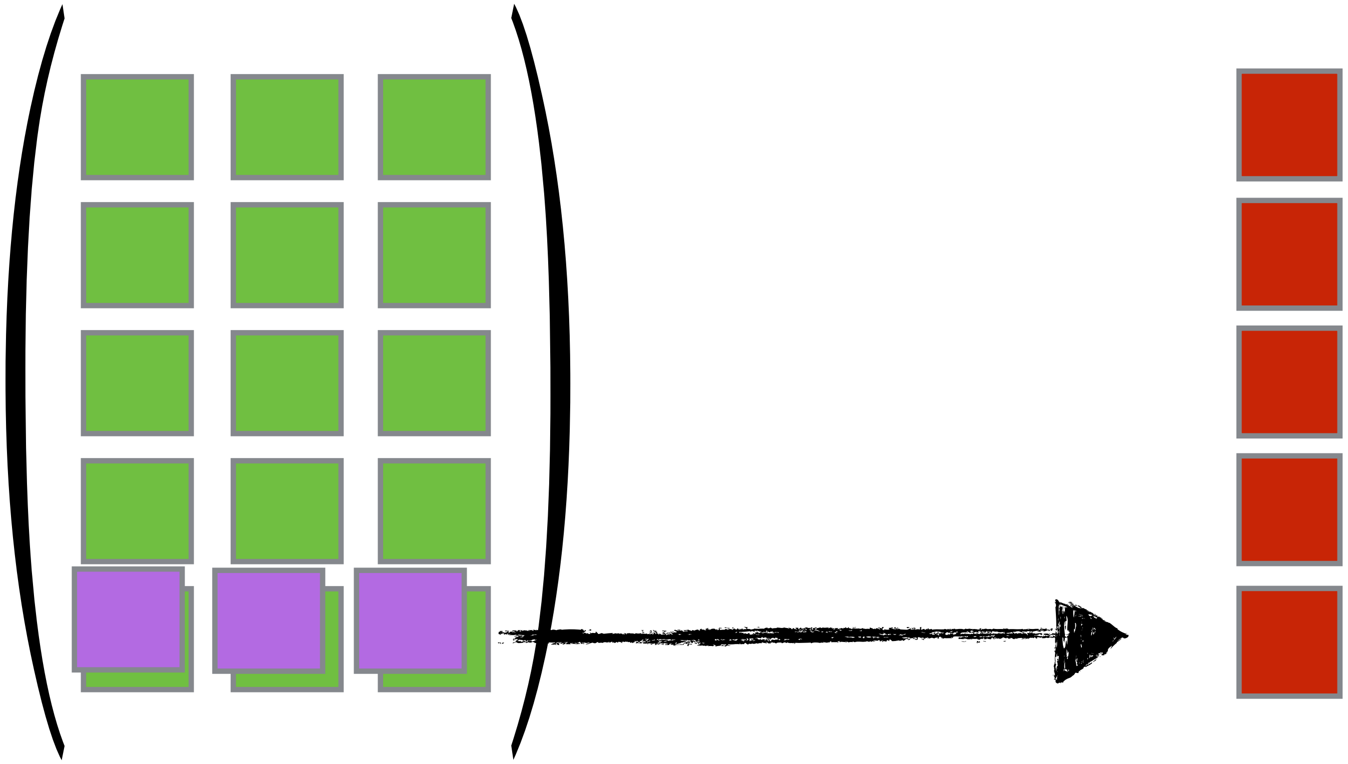


Sizes must match up!

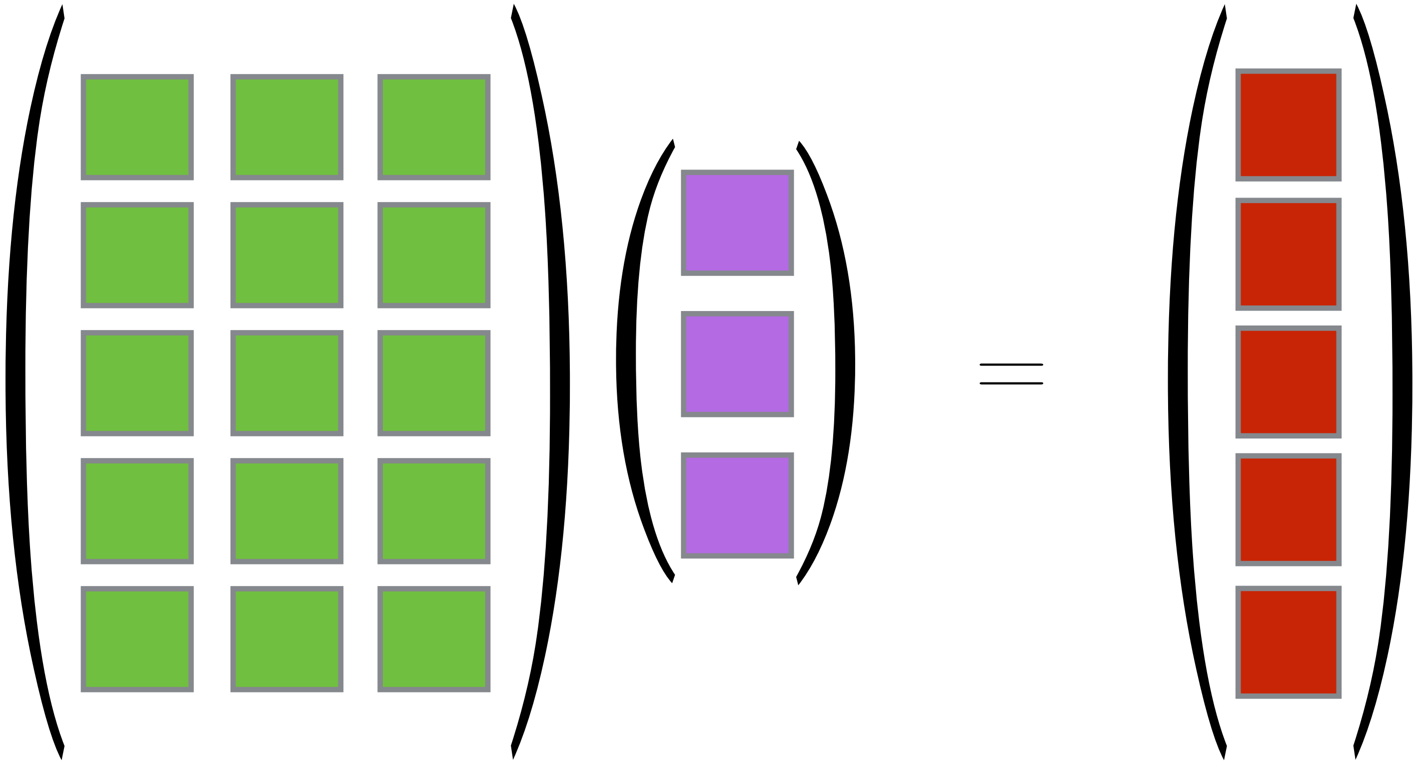
Matrix-Vector Multiplication (I-P view)



Matrix-Vector Multiplication (I-P view)



Matrix-Vector Multiplication (I-P view)



Example: Matrix-Vector Products

Let

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \mathbf{q} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

Determine whether the following matrix-vector products are possible. When possible, compute the product.

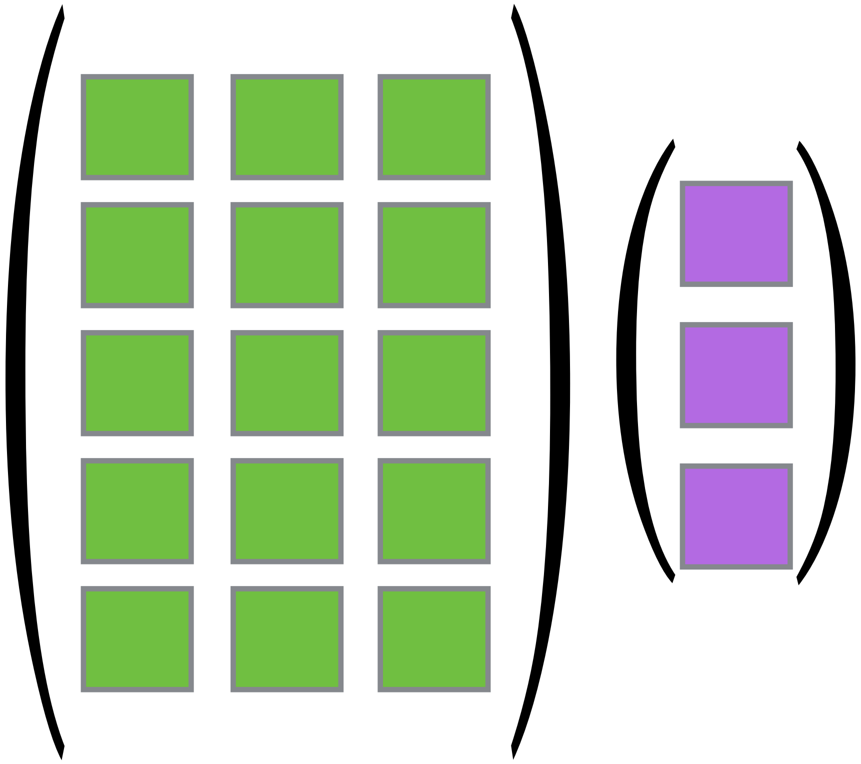
a. $\mathbf{Aq}$

b. $\mathbf{Av}$

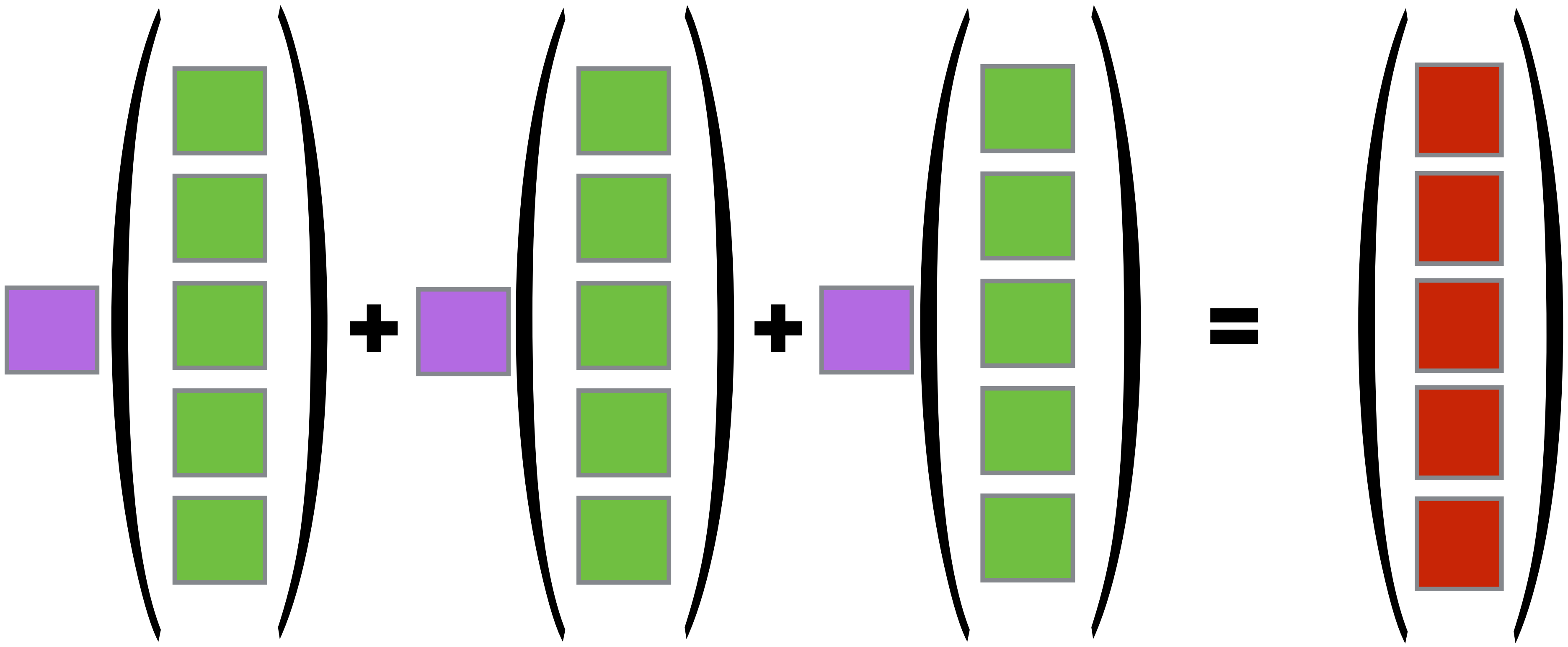
Matrix-Vector Multiplication

Linear Combination View (L-C View)

Matrix-Vector Multiplication (L-C view)



Matrix-Vector Multiplication (L-C view)



Example: Linear Combination View

$$\mathbf{Av} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2(3) + 3(2) \\ -1(3) + 4(2) \\ 5(3) + 1(2) \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Example: Linear Combination View

$$\mathbf{Av} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2(3) + 3(2) \\ -1(3) + 4(2) \\ 5(3) + 1(2) \end{pmatrix}$$

$$= 3 \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

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Matrix-Matrix Multiplication

- Matrix multiplication is NOT commutative.

$$AB \neq BA$$

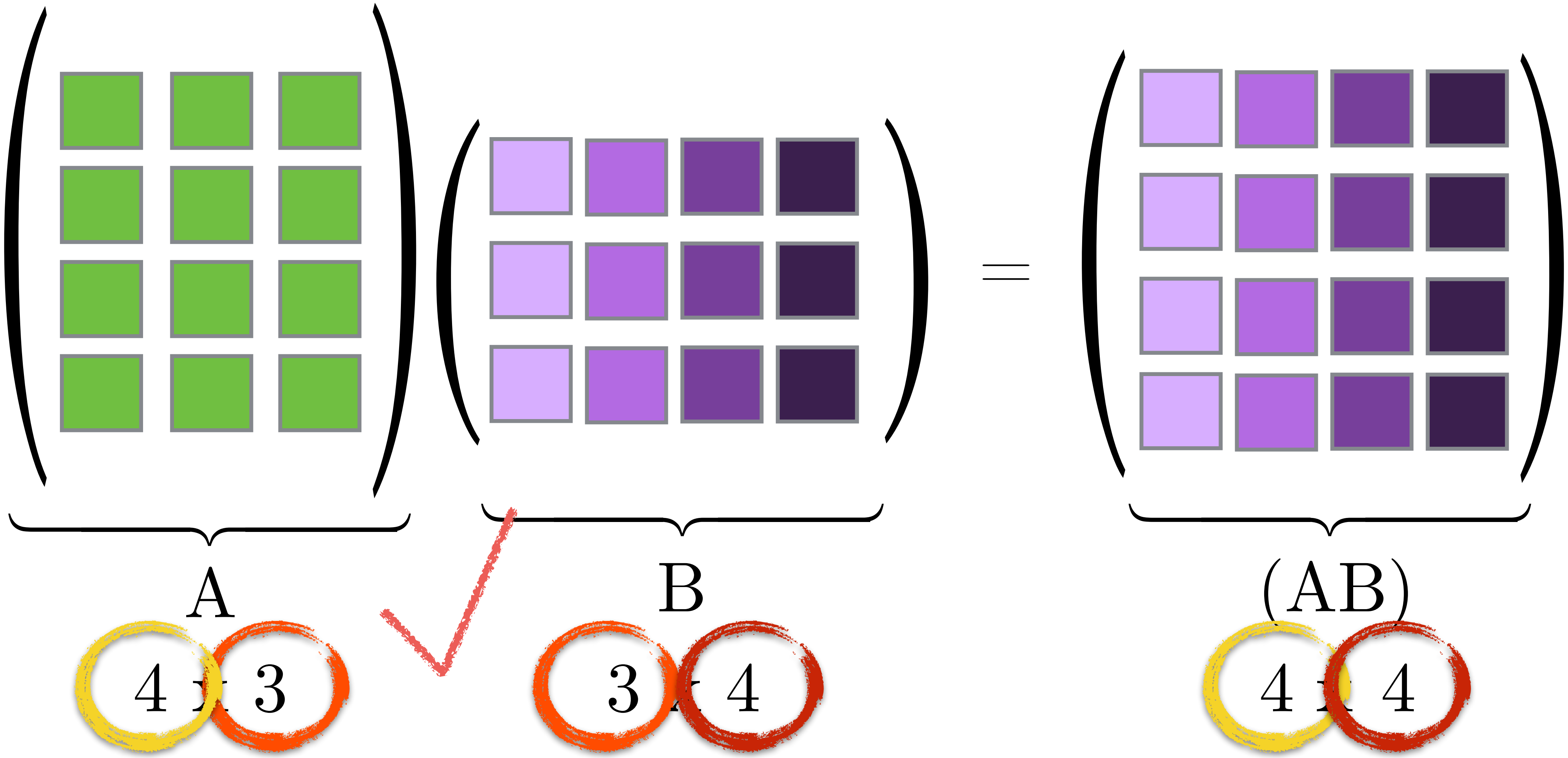
- Matrix multiplication is only defined for dimension-compatible matrices

Matrix-Matrix Multiplication (I-P View)

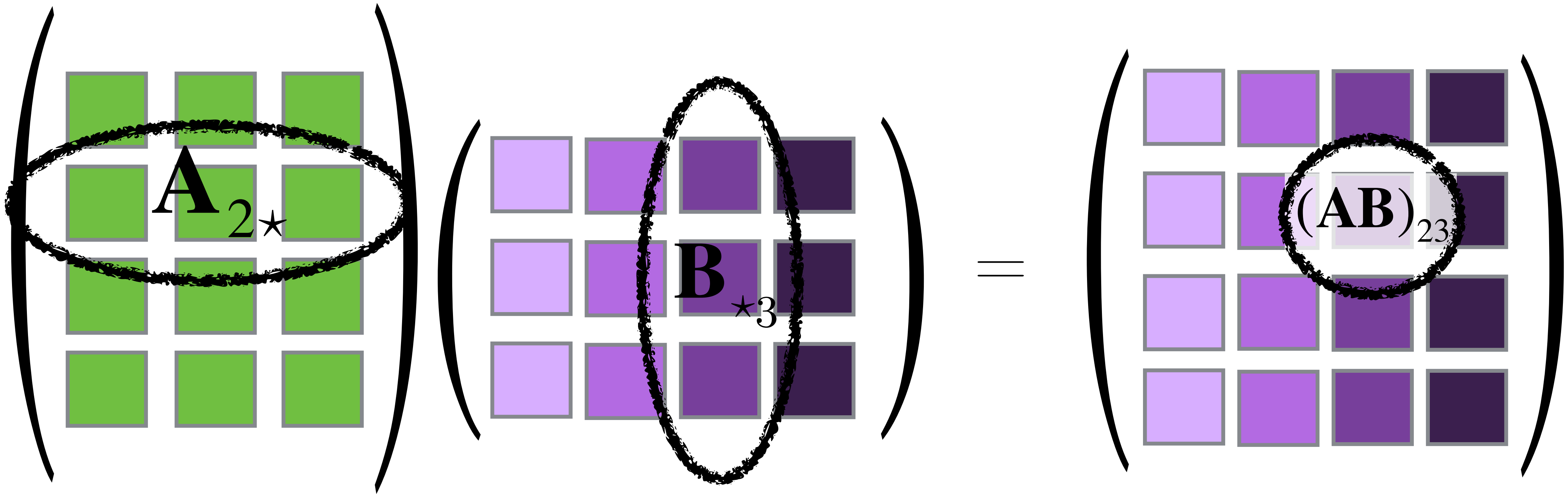
- If A and B are dimension compatible, then we compute the product AB by multiplying every row of A by every column of B *(inner products)*.
- The $(i,j)^{\text{th}}$ entry of the product AB is the i^{th} row of A multiplied by the j^{th} column of B

Matrix-Matrix Multiplication (I-P View)

A and B are *dimension compatible for the product* AB if the number of columns in A is equal to the number of rows in B



Matrix-Matrix Multiplication (I-P View)



$$\boxed{(\mathbf{AB})_{ij} = \mathbf{A}_{i\star}\mathbf{B}_{\star j}}$$

Example: Matrix-Matrix Multiplication

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & -2 \\ 2 & -3 \end{pmatrix}$$

Check Your Understanding

Suppose we have

$$\mathbf{A}_{4\times 6} \quad \mathbf{B}_{5\times 5} \quad \mathbf{M}_{5\times 4} \quad \mathbf{P}_{6\times 5}$$

Circle the matrix products that are possible to compute and write the dimension of the result.

$$\mathbf{AM} \quad \mathbf{MA} \quad \mathbf{BM} \quad \mathbf{MB} \quad \mathbf{PA} \quad \mathbf{PM} \quad \mathbf{AP} \quad \mathbf{A}^T\mathbf{P} \quad \mathbf{M}^T\mathbf{B}$$

Check your Understanding SOLUTION

Suppose we have

$$\mathbf{A}_{4\times 6} \quad \mathbf{B}_{5\times 5} \quad \mathbf{M}_{5\times 4} \quad \mathbf{P}_{6\times 5}$$

Circle the matrix products that are possible to compute and write the dimension of the result.

$\mathbf{AM}$	[✓ $\mathbf{MA}$]	[✓ $\mathbf{BM}$]	$\mathbf{MB}$	$\mathbf{PA}$	[✓ $\mathbf{PM}$]	[✓ $\mathbf{AP}$]	$\mathbf{A}^T\mathbf{P}$	[✓ $\mathbf{M}^T\mathbf{B}$]
	5×6	5×4			6×4	4×5		4×5

Check Your Understanding

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} -2 & 1 & -1 & 2 & -2 \\ 1 & -2 & 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

Determine the following matrix products, if possible.

a. $\mathbf{AC}$

b. $\mathbf{AM}$

c. $\mathbf{AC}^T$

Check your Understanding SOLUTION

Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} -2 & 1 & -1 & 2 & -2 \\ 1 & -2 & 0 & -1 & 2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

Determine the following matrix products, if possible.

a. $\mathbf{AC} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$

b. $\mathbf{AM} = \begin{pmatrix} -1 & -1 & -1 & 1 & 0 \\ 1 & -2 & 0 & -1 & 2 \end{pmatrix}$

c. $\mathbf{AC}^T =$ not possible

NOT Commutative

- Very important to remember that

Matrix multiplication is NOT commutative!

- As we see in previous exercise, common to be able to compute product AB when the reverse product, BA, is not even defined.

- Even when both products are possible, almost *never* the case that AB = BA.

Diagonal Scaling

Multiplication by a diagonal matrix

Multiplication by a diagonal matrix

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \enclose{circle}{3} & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

The net effect is that the rows of A are scaled by the corresponding diagonal element of D

Multiplication by a diagonal matrix

Rather than computing DA, what if we instead put the diagonal matrix on the right hand side and compute AD?

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\mathbf{AD} =$$

(Exercise)

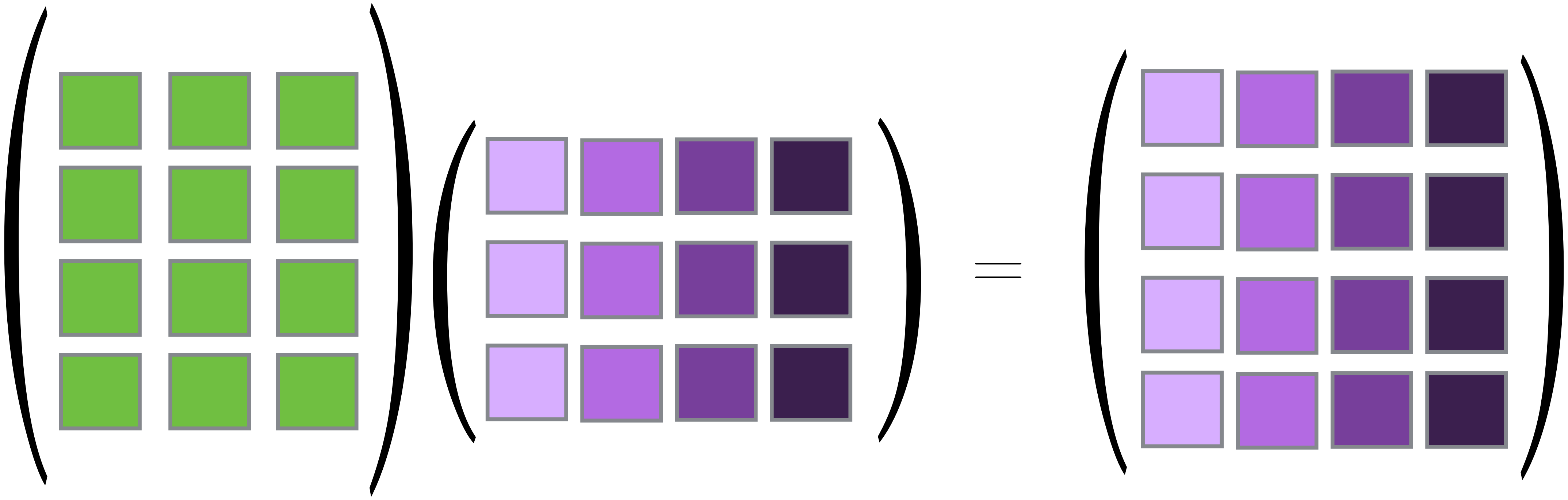
Matrix-Matrix Multiplication

As a Collection of Linear Combinations (L-C View)

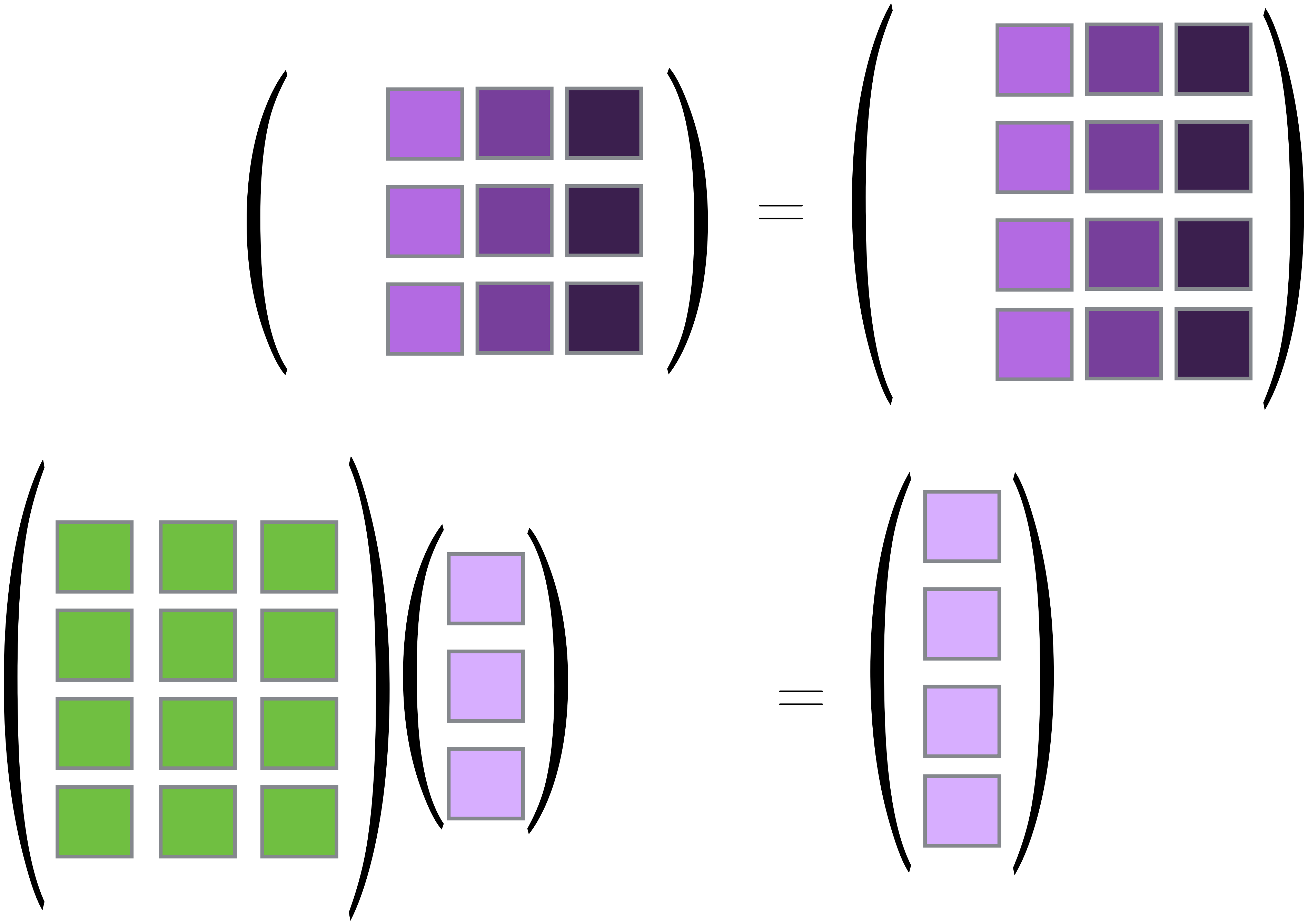
Matrix-Matrix Multiplication (L-C View)

- Just a collection of matrix-vector products (linear combinations) with different coefficients.
- Each linear combination involves the same set of vectors (the green columns) with different coefficients (the purple columns).

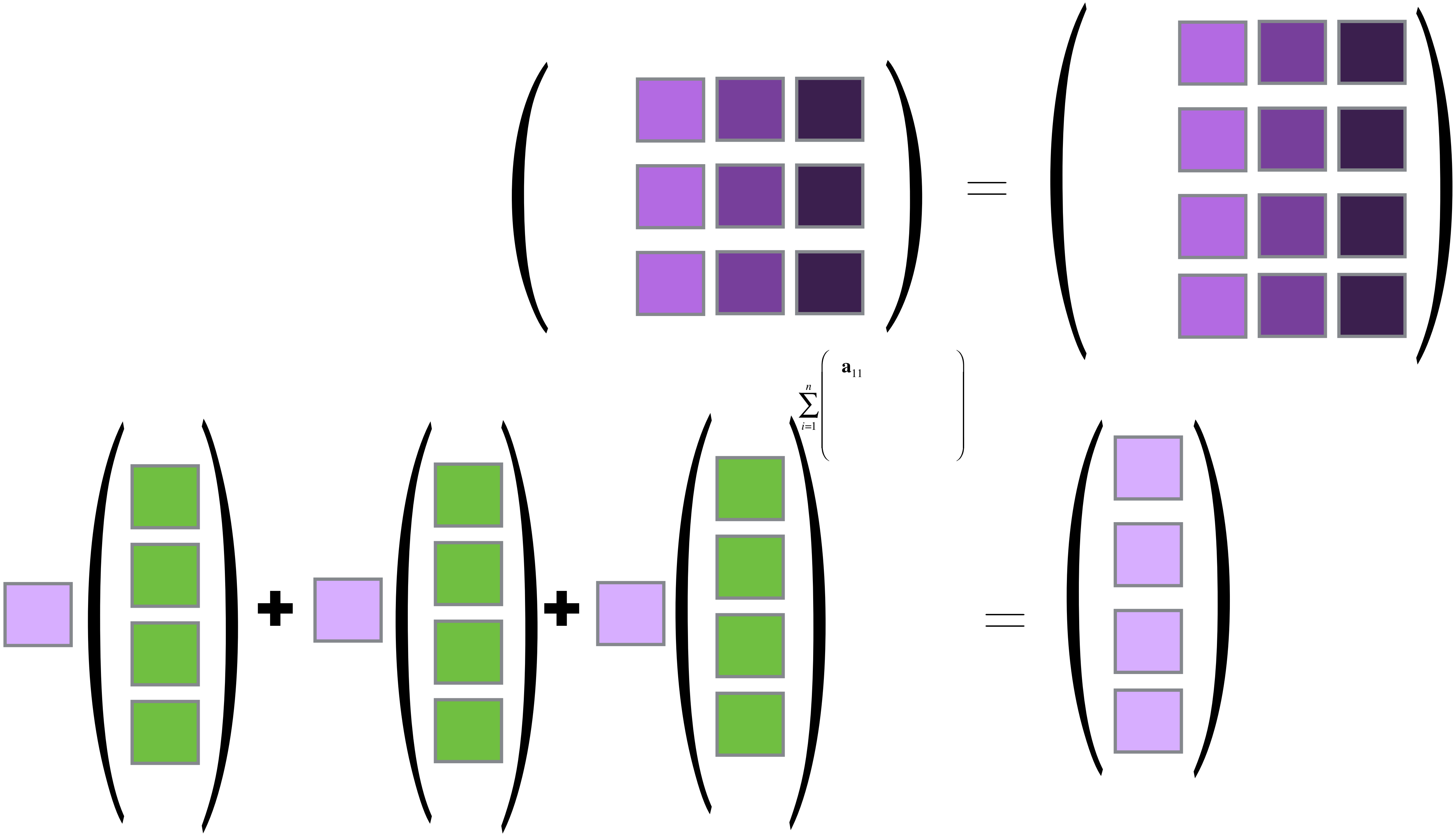
Matrix-Matrix Multiplication (L-C View)



Matrix-Matrix Multiplication (L-C View)

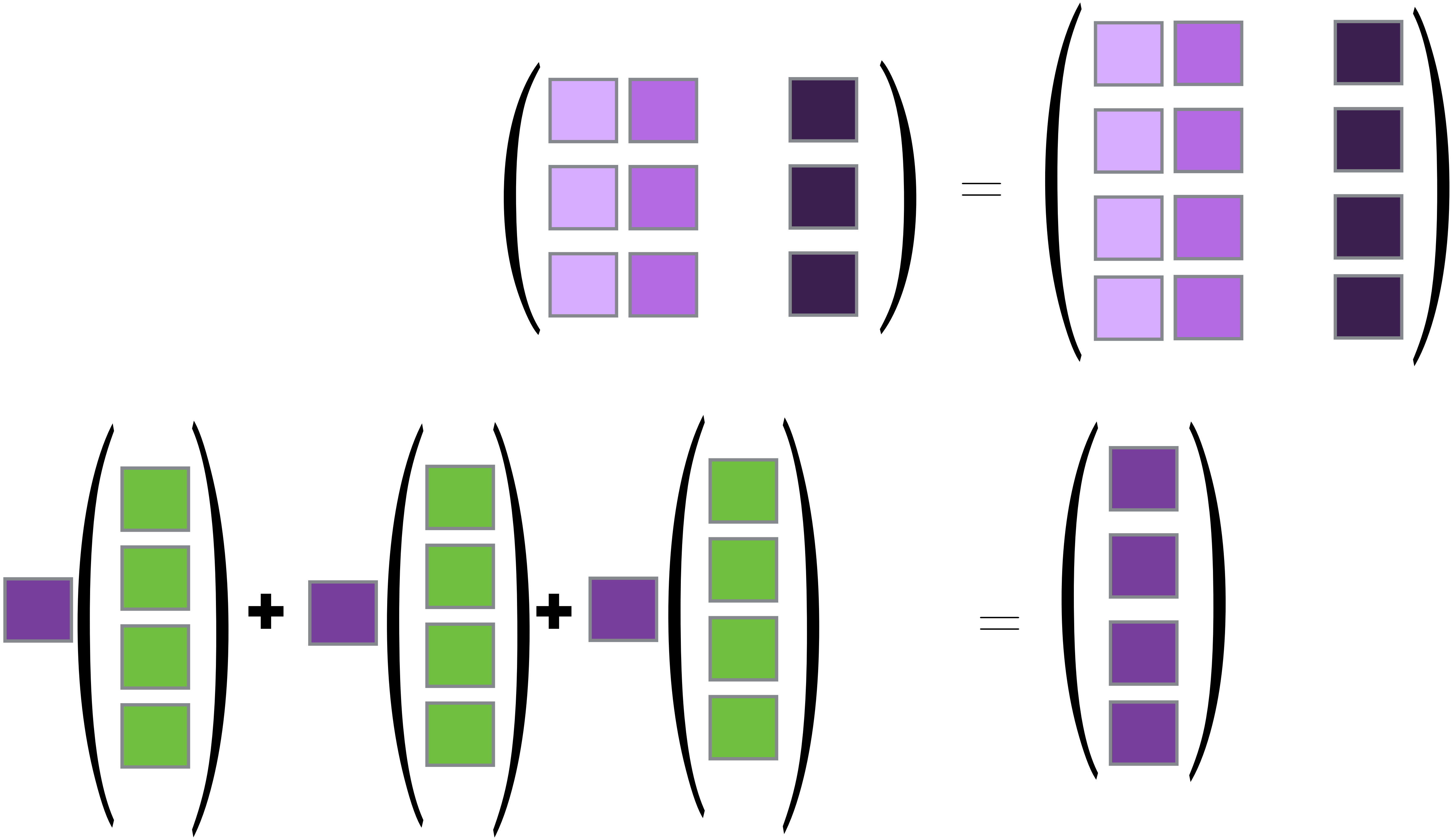


Matrix-Matrix Multiplication (L-C View)

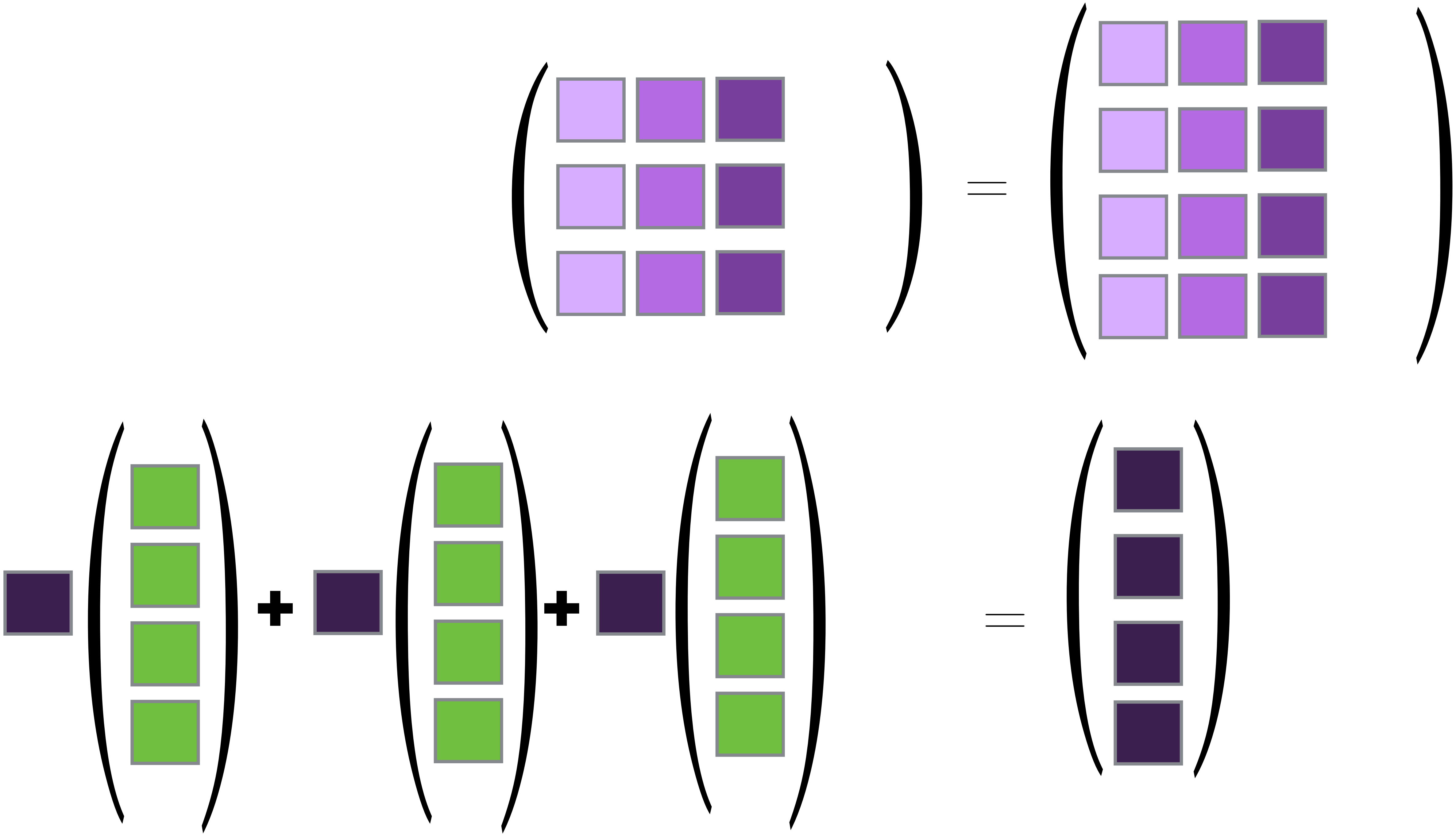


$$\sum_{i=1}^{n} \begin{pmatrix} \mathbf{a}_{11} & \end{pmatrix}$$

Matrix-Matrix Multiplication (L-C View)



Matrix-Matrix Multiplication (L-C View)



Matrix-Matrix Multiplication (L-C View)

- Just a collection of matrix-vector products (linear combinations) with different coefficients.
- Each linear combination involves the same set of vectors (the green columns) with different coefficients (the purple columns).
- This has important implications!

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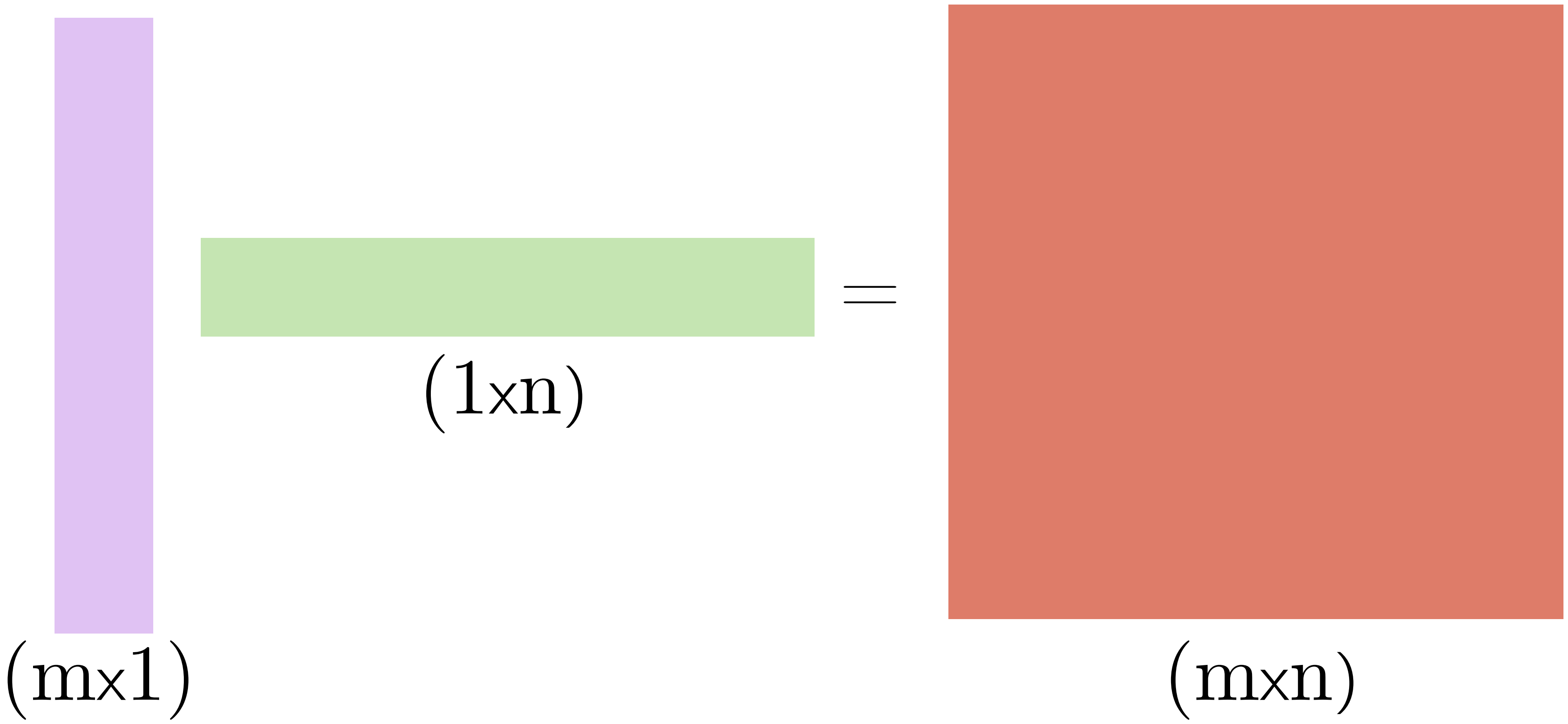
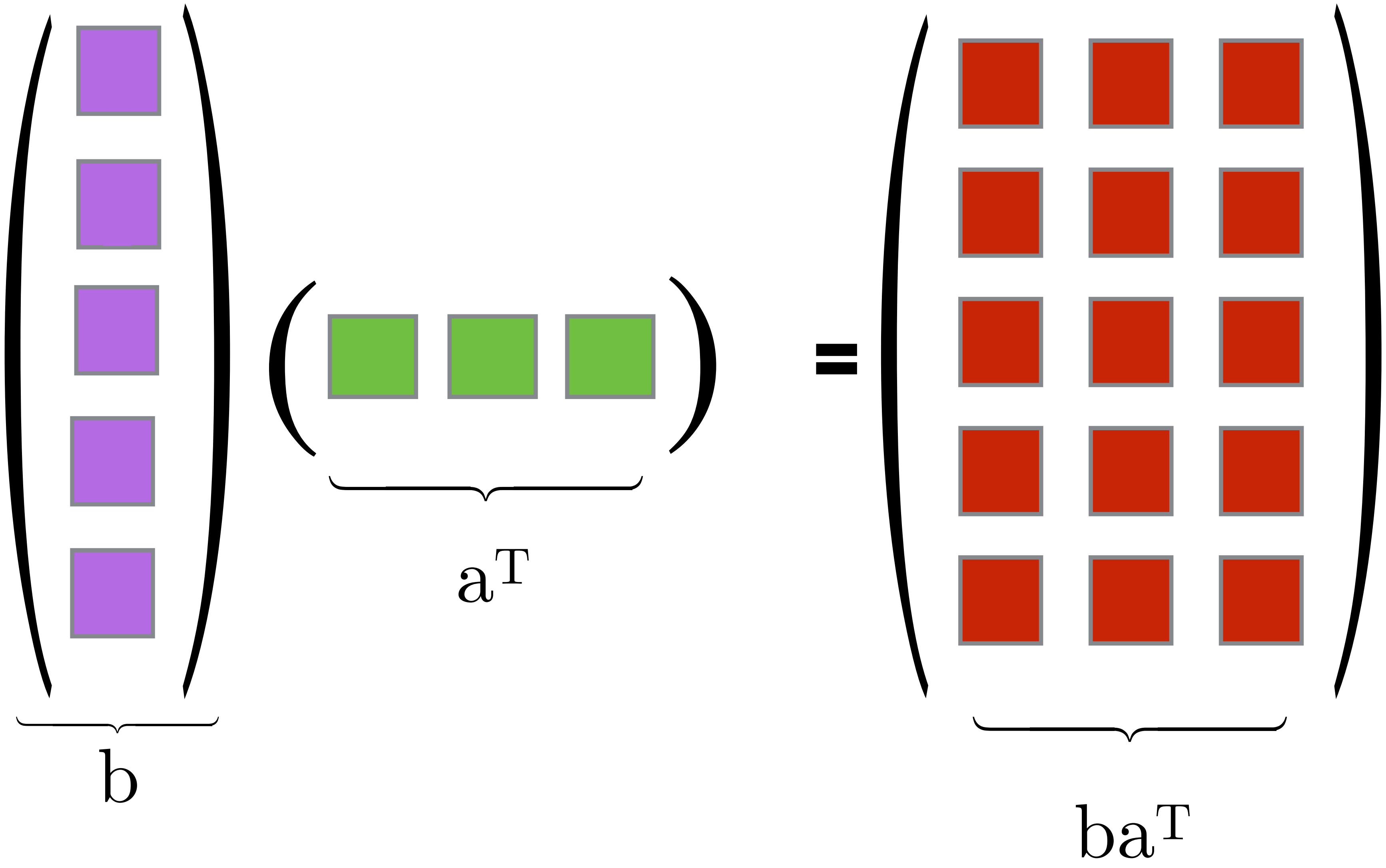
Vector Outer Product

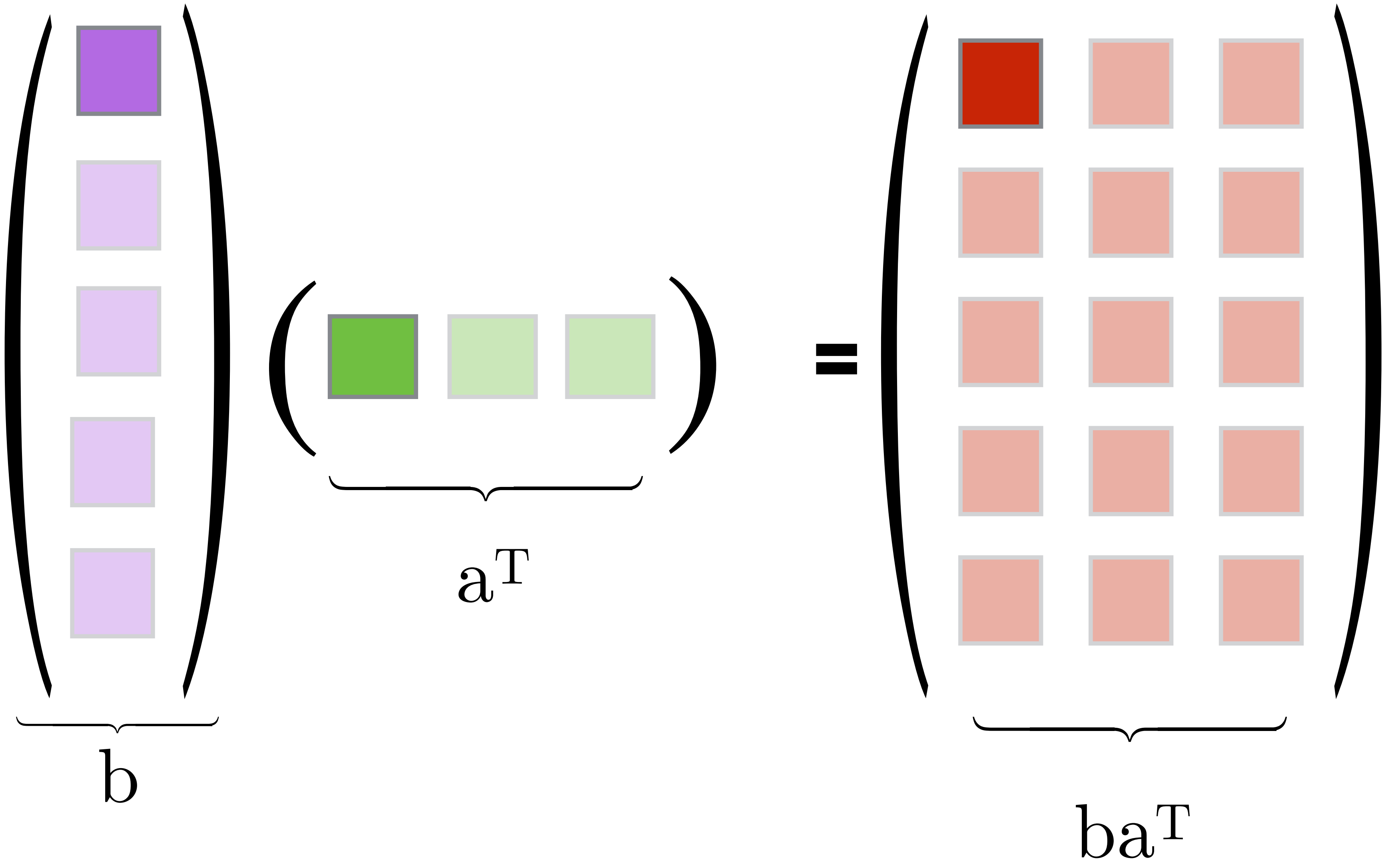
- The vector outer product is the multiplication of a column vector times a row vector.
- For any column/row this product is possible
- The result of this product is a *matrix*!



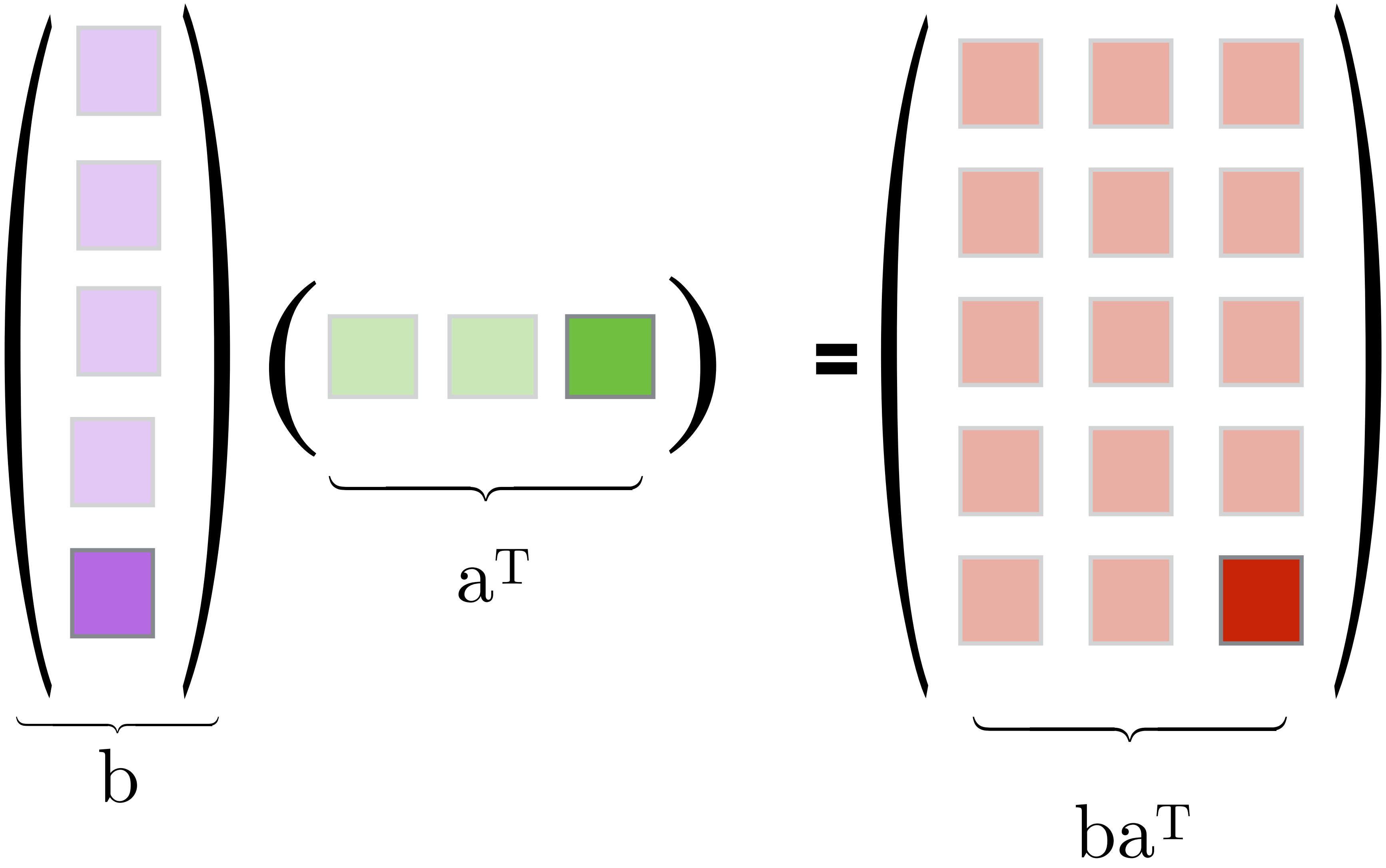
Outer Product (column x row)



Outer Product (column x row)



Outer Product (column x row)



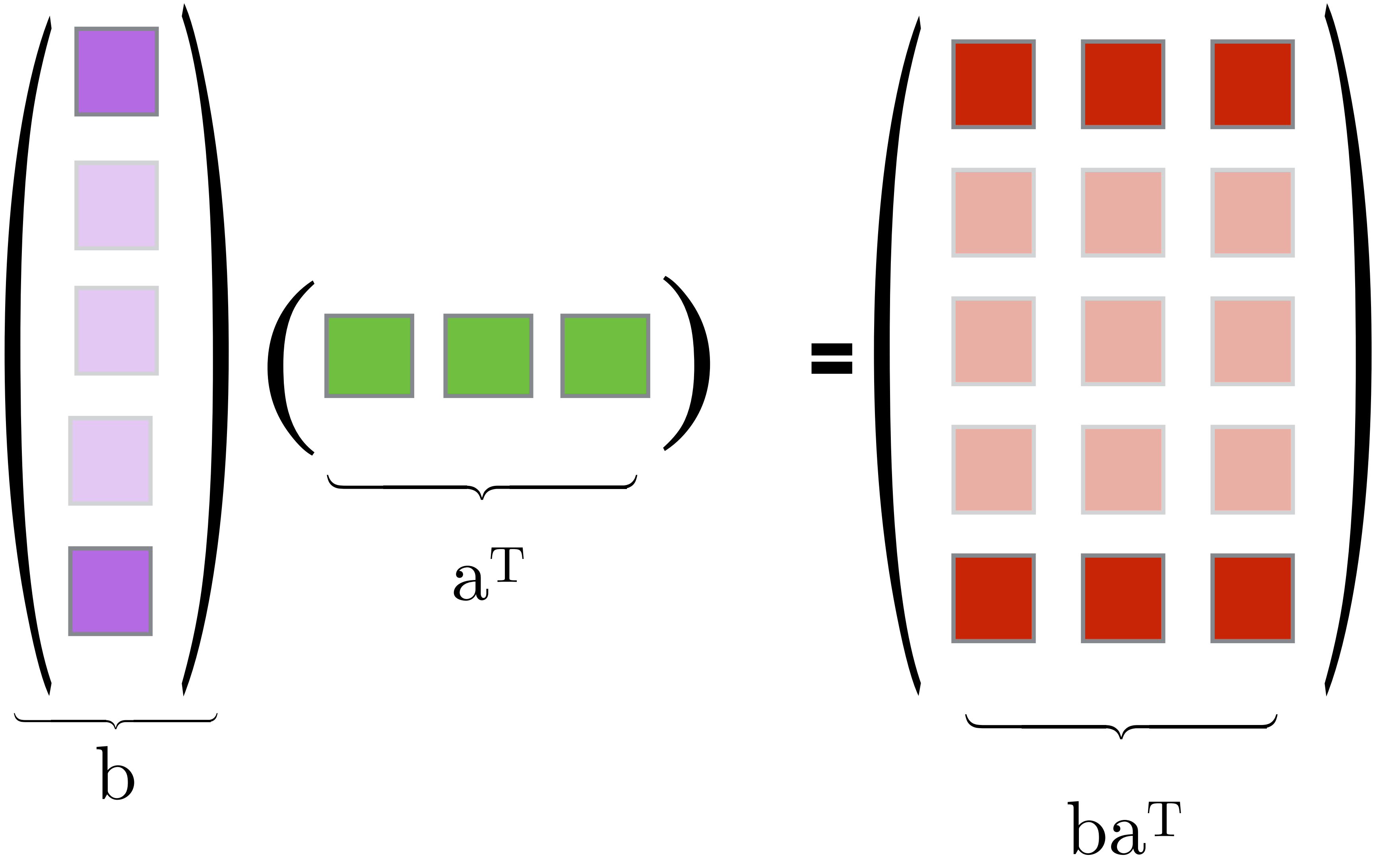
Example: Outer Product

Let $\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$. Then,

$$\mathbf{x}\mathbf{y}^T = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \begin{pmatrix} 1 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 15 & 9 \\ 4 & 20 & 12 \\ -2 & -10 & -6 \end{pmatrix}$$

Outer Product has rank 1

From the previous example, you can see that the rows of an outer product are necessarily multiples of each other.



Matrix-Matrix Multiplication

As a Sum of Outer Products (O-P View)

Matrix-Matrix Multiplication (O-P View)

We can write the product AB as a *sum of outer products* of columns of $A_{(mxn)}$ and rows of $B_{(nxp)}$

$$\mathbf{AB} = \sum_{i=1}^{n} \mathbf{A}_{\star i}\mathbf{B}_{i\star}$$

This view decomposes the product AB into the sum of n matrices, each of which has rank 1 (discussed later).

Challenge Puzzle

- Suppose we have 1,000 individuals that have been divided into 5 different groups each year for 20 years.

- We need to make a 1000x1000 matrix C where

$$\boxed{\mathrm{C}_{\mathrm{ij}} = \# \text{ times person } i \text{ grouped with person } j}$$

- The data currently has 1000 rows and 5x20 = 100 binary columns indicating whether each individual was a member of each group (yLgK: yearLgroupK): (y1g1, y1g2, y1g3, y1g4, y1g5, y2g1, ... y20g5)

- Can we use what we've just learned to help us here?

Special Cases of Matrix Multiplication

The Identity and the Inverse

The Identity Matrix

- The identity matrix, ‘I’, is to matrices what the number ‘1’ is to scalars.
- It is the multiplicative identity.

$$\mathbf{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- For any matrix (or vector) A, multiplying A by the (appropriately sized) identity matrix on either side does not change A:

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

The Matrix Inverse

- For certain square matrices, A, an inverse matrix written A^{-1}, exists such that:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

  - A MATRIX MUST BE SQUARE TO HAVE AN INVERSE.
  - NOT ALL SQUARE MATRICES HAVE AN INVERSE
  - Only full-rank, square matrices are invertible. (more on this later)
- For now, understand that the inverse matrix serves like the multiplicative inverse in scalar algebra:

$$(2)(2^{-1}) = (2)(\frac{1}{2}) = 1$$

- Multiplying a matrix by its inverse (if it exists) yields the multiplicative identity, I

The Matrix Inverse

- A *square* matrix which has an inverse is equivalently called:
    - Non-singular
    - Invertible
    - Full Rank

Don't Cancel That!!

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

Cancellation implies inversion

Canceling numbers in scalar algebra:

$$2x = 2y$$

$$\cancel{2}x = \cancel{2}y$$

$$\frac{1}{2}2x = \frac{1}{2}2y$$

$$1x = 1y$$

Canceling matrices in linear algebra:

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Inverses can help
solve an equation...
WHEN THEY EXIST!