Chapter 4

Extra Examples: Solving Systems + Determining Rank

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{array}\right) \xrightarrow{R2'=R2-2R1} \left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{array}\right) \xrightarrow{R3'=R3-2R1} \left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ -1 & 1 & 3 & 6 & 5 \end{array}\right)$$

Example 1

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ -1 & 1 & 3 & 6 & 5 \end{array}\right) \xrightarrow{R4'=R4+R1} \left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{array}\right) \xrightarrow{R2\leftrightarrow R3} \left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{array}\right)$$

Example 1

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{array}\right) \xrightarrow{R4'=R4+3R2} \left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & -5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & -5 \end{array}\right) \xrightarrow{R3'=\frac{1}{8}R3} \left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & -5 \end{array}\right)$$

Example 1

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & -5 \end{array}\right) \xrightarrow{R4'=R4-29R3} \left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{array}\right)$$

Inconsistent

Example 1

$$\left(\begin{array}{cccc|c} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{array}\right)$$

3 Pivots $\Longrightarrow$ Rank of coefficient matrix is 3

(NOT full rank)

Example 2

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right) \xrightarrow[R3'=R3+R1]{R2'=R2-2R1} \left(\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right) \xrightarrow{R3'=R3+2R2} \left(\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Example 2

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R1'=R1+R2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced Row Echelon Form

2 Pivots $\Longrightarrow$ Rank of coefficient matrix is 2

Example 2

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

- 2 free variables, x_2 and x_4
- Next, write the pivot column variables in terms of free variables:

$$\begin{aligned} x_1 - x_2 + x_4 &= 2 \\ x_3 - x_4 &= 1 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} x_1 &= x_2 - x_4 + 2 \\ x_3 &= x_4 + 1 \end{aligned}$$

Example 2

$$\begin{aligned} x_1 &= x_2 - x_4 + 2 \\ x_3 &= x_4 + 1 \end{aligned}$$

Let $x_2 = s$ and $x_4 = t$. Then we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} s - t + 2 \\ s \\ t + 1 \\ t \end{pmatrix}$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$