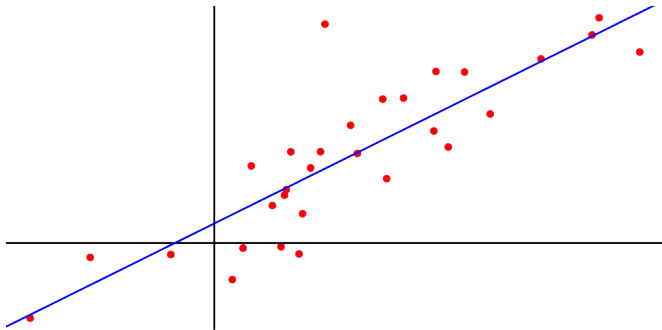# Chapter 1
What is Linear Algebra?

What is Linear Algebra?

The study of **linear** functions.
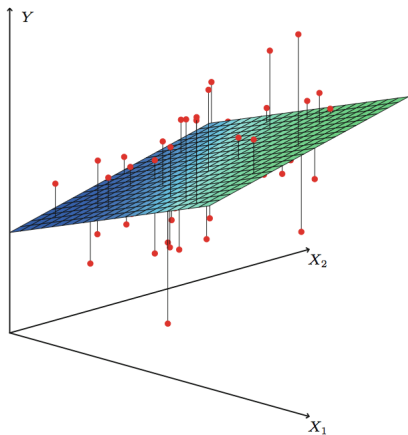The word **linear** means *straight* or *flat*.

$$y = \beta_0 + \beta_1 x$$

Linear functions involve only addition and scalar multiplication.

Higher Dimensions

- In the real world, our regression equations and modeling problems often involve more than 2 variables.
- With 3 variables, our linear (*flat*) regression equation creates a plane.

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

Higher Dimensions

- When we have more than 3 variables, we can no longer imagine the regression surface. However, because it is linear we know that it is "flat". It does not bend or curve.

- In linear algebra we will not see equations or functions like:

$$x^2 + y^2 + 3z = 10 \ \text{ or } \ 2xy + \sqrt{z} + \frac{1}{x} = 1 \ \text{ or } \ log(x - y) + e^{2z}$$

These functions are *nonlinear.*

- A linear function involves *only* scalar multiplication and addition/subtraction, for example:

$$2x - 3y + z = 9 \ \text{ or } \ 4x_1 - 3x_2 + 9x_3 + x_4 - x_5 + 2x_6 = 2$$

- Linear algebra involves the study of **matrices** and **vectors**. These objects are at the core of almost every data problem that exists.

Matrices, Vectors and Scalars

- A **matrix** is an array of numbers, logically ordered by rows and columns, for example:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 0 \end{pmatrix} \qquad \mathbf{H} = \begin{pmatrix} 6 & 5 & 10 \\ 0.1 & 0.5 & 0.9 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

- A **vector** is a matrix with only one row or column, for example:

$$\mathbf{x} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 3 & 5 & 1 & 0 & 2 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

- A **scalar** is simply a number, for example:

$$\pi, \ 2, \ 4, \ \sqrt{6}, \ 10^{16}, \ \frac{1}{7}$$

Matrices, Vectors and Scalars

- To numerically work with data, any system will turn that data into a matrix:

Name	Credit Score	Income
John	780	95000
Sam	600	60000
Elena	550	65000
Jim	400	35000
Eric	450	40000
Helen	750	80000

$$\begin{array}{c} \\ John \\ Sam \\ Elena \\ Jim \\ Eric \\ Helen \end{array} \begin{array}{cc} CreditScore & Income \\ \left(\begin{array}{cc} 780 & 95000 \\ 600 & 60000 \\ 550 & 65000 \\ 400 & 35000 \\ 450 & 40000 \\ 750 & 80000 \end{array}\right) \end{array}$$

Before we can really begin to talk about the arithmetic of matrices and vectors, it is *very* important that we know how to describe them.

- Size or Dimension of a matrix
- (i,j)-notation
- Notation
- Transpose and Symmetry
- Special Matrices

Describing Matrices and Vectors

- Size or Dimension of a matrix
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Size/Dimension of a matrix

This is merely the number of rows and columns in the matrix.
The number of rows is *always* specified first.

An $m \times n$ matrix has m rows and n columns.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 0 \end{pmatrix} \qquad \mathbf{H} = \begin{pmatrix} 6 & 5 & 10 \\ 0.1 & 0.5 & 0.9 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

- **A** is 3×2
- **H** is 4×3
- We can write $\mathbf{A}_{3\times 2}$ and $\mathbf{H}_{4\times 3}$ to specify the size.

Square vs. Rectangular Matrix

A **square** matrix is a matrix that has the same number of rows as columns.

<p style="text-align:center; color:blue">An $n \times n$ matrix is square.</p>

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 7 \\ 3 & 5 & 1 \\ 4 & 0 & 2 \end{pmatrix} \qquad \mathbf{H} = \begin{pmatrix} 6 & 5 & 10 \\ 0.1 & 0.5 & 0.9 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

- **A** is **square**. $\mathbf{A}_{3\times 3}$
- **H** is **rectangular** $\mathbf{H}_{4\times 3}$

A **vector** is just a matrix with one row or column. We will often specify directly whether the vector is a row or column.

- A **row vector** with n entries is a $1 \times n$ matrix.

$$\mathbf{t} = \begin{pmatrix} t_1 & t_2 & \dots & t_n \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 3 & 5 & 1 & 0 & 2 \end{pmatrix}$$

- A **column vector** with n entries is a $n \times 1$ matrix.

$$\mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix}$$

a) For the following matrices, determine the dimensions:

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \\ 8 & 5 & 0.2 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} .01 & .5 & 1.6 & 1.7 \\ .1 & 3.5 & 4 & 2 \\ .61 & .55 & .46 & .17 \\ 1.2 & 1.5 & 1.6 & 1 \\ .31 & .35 & 1.3 & 2.3 \\ 2.3 & 3.5 & .06 & .7 \\ .3 & .2 & 2.1 & 1.8 \end{pmatrix} \quad \mathbf{t} = \begin{pmatrix} 1 \\ 1.3 \\ 0.8 \\ 2 \\ 2.5 \\ 0.8 \\ 0.9 \end{pmatrix}$$

b) Give an example of a square matrix.

a) For the following matrices, determine the dimensions:

$$\underset{4\times3}{\mathbf{B}} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \\ 8 & 5 & 0.2 \end{pmatrix} \quad \underset{7\times4}{\mathbf{C}} = \begin{pmatrix} .01 & .5 & 1.6 & 1.7 \\ .1 & 3.5 & 4 & 2 \\ .61 & .55 & .46 & .17 \\ 1.2 & 1.5 & 1.6 & 1 \\ .31 & .35 & 1.3 & 2.3 \\ 2.3 & 3.5 & .06 & .7 \\ .3 & .2 & 2.1 & 1.8 \end{pmatrix} \quad \underset{7\times1}{\mathbf{t}} = \begin{pmatrix} 1 \\ 1.3 \\ 0.8 \\ 2 \\ 2.5 \\ 0.8 \\ 0.9 \end{pmatrix}$$

b) A square matrix has the same number of rows and columns, for example:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Describing Matrices and Vectors

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- The element of matrix **A** found in *row i* and *column j* is written

$$A_{ij} \quad \text{or sometimes } a_{ij}$$

- The **diagonal** elements of a square matrix are those that have identical row and column indices: $\mathbf{A}_{ii}$
- To refer to the ith row of **A** we will use the notation $\mathbf{A}_{i\star}$
- Similarly, to refer to the jth column of **A** we will use the notation $\mathbf{A}_{\star j}$.

(i,j) - Notation

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} .01 & .5 & 1.6 & 1.7 \\ .1 & 3.5 & 4 & 2 \\ .61 & .55 & .46 & .17 \\ 1.2 & 1.5 & 1.6 & 1 \\ .31 & .35 & 1.3 & 2.3 \end{pmatrix}$$

- $\mathbf{B}_{31} = 3$
-
-

(i,j) - Notation

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} .01 & .5 & 1.6 & 1.7 \\ .1 & 3.5 & 4 & 2 \\ .61 & .55 & .46 & .17 \\ 1.2 & 1.5 & 1.6 & 1 \\ .31 & .35 & 1.3 & 2.3 \end{pmatrix}$$

- $\mathbf{B}_{31} = 3$
- $\mathbf{C}_{23} = 4$
-

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} .01 & .5 & 1.6 & 1.7 \\ .1 & 3.5 & 4 & 2 \\ .61 & .55 & .46 & .17 \\ 1.2 & 1.5 & 1.6 & 1 \\ .31 & .35 & 1.3 & 2.3 \end{pmatrix}$$

- $\mathbf{B}_{31} = 3$
- $\mathbf{C}_{23} = 4$
- $\mathbf{B}_{11} = 1$
    - **Diagonal elements** of a square matrix have the same row and column index ($\mathbf{B}_{11}$, $\mathbf{B}_{22}$, $\mathbf{B}_{33}$).

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} .01 & .5 & 1.6 & 1.7 \\ .1 & 3.5 & 4 & 2 \\ .61 & .55 & .46 & .17 \\ 1.2 & 1.5 & 1.6 & 1 \\ .31 & .35 & 1.3 & 2.3 \end{pmatrix}$$

- $\mathbf{B}_{\star 1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

- $\mathbf{C}_{3\star} = \begin{pmatrix} .61 & .55 & .46 & .17 \end{pmatrix}$

When it comes to vectors, we no longer need two subscripts because vectors have only one row or one column. Thus, we can use a single subscript to reference the element we want:

$\mathbf{v}_i$ is the ith element in a vector $\mathbf{v}$.

$$\mathbf{v} = \begin{pmatrix} 4 \\ 2 \\ 7 \\ 1 \\ 3 \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} 0.25 & 0.3 & 0.15 & 0.3 \end{pmatrix}$$

$$v_3 = 7 \qquad v_5 = 3 \qquad p_2 = 0.3$$

For the following matrices, give the elements or row/column vectors listed below:

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} .01 & .5 & 1.6 & 1.7 \\ .1 & 3.5 & 4 & 2 \\ .61 & .55 & .46 & .17 \\ 1.2 & 1.5 & 1.6 & 1 \\ .31 & .35 & 1.3 & 2.3 \\ 2.3 & 3.5 & .06 & .7 \\ .3 & .2 & 2.1 & 1.8 \end{pmatrix} \quad \mathbf{t} = \begin{pmatrix} 1 \\ 1.3 \\ 0.8 \\ 2 \\ 2.5 \\ 0.8 \\ 0.9 \end{pmatrix}$$

$\mathbf{B}_{13} = \qquad \mathbf{B}_{\star 2} = \qquad \mathbf{C}_{51} = \qquad \mathbf{C}_{3\star} = \qquad t_6 =$

For the following matrices, give the elements or row/column vectors listed below:

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} .01 & .5 & 1.6 & 1.7 \\ .1 & 3.5 & 4 & 2 \\ .61 & .55 & .46 & .17 \\ 1.2 & 1.5 & 1.6 & 1 \\ .31 & .35 & 1.3 & 2.3 \\ 2.3 & 3.5 & .06 & .7 \\ .3 & .2 & 2.1 & 1.8 \end{pmatrix} \quad \mathbf{t} = \begin{pmatrix} 1 \\ 1.3 \\ 0.8 \\ 2 \\ 2.5 \\ 0.8 \\ 0.9 \end{pmatrix}$$

$\mathbf{B}_{13} = 0$ $\mathbf{B}_{\star 2} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $\mathbf{C}_{51} = 0.31$ $\mathbf{C}_{3\star} = \begin{pmatrix} .61 & .55 & .46 & .17 \end{pmatrix}$ $\mathbf{t}_6 = 0.8$

Defining a matrix by (i,j) Notation

The nice thing about (i,j) notation is that it can help us to define an entire matrix. Consider the following data, where 6 different students are assigned to teams over the course of a school year.

Summer Teams		Fall Teams		Spring Teams	
Team 1	Team 2	Team 1	Team 2	Team 1	Team 2
Student 1	Student 4	Student 2	Student 5	Student 2	Student 3
Student 2	Student 5	Student 3	Student 1	Student 4	Student 5
Student 3	Student 6	Student 4	Student 6	Student 1	Student 6

We could define a matrix, **M**, to represent this data by defining each element $\mathbf{M}_{ij}$ as follows:

$$\mathbf{M}_{ij} = \left\{ \begin{array}{l} \text{\# of times Student i has worked with Student j if } i \neq j \\ 0 \text{ if } i = j \end{array} \right.$$

Summer Teams		Fall Teams		Spring Teams	
Team 1	Team 2	Team 1	Team 2	Team 1	Team 2
Student 1	Student 4	Student 2	Student 5	Student 2	Student 3
Student 2	Student 5	Student 3	Student 1	Student 4	Student 5
Student 3	Student 6	Student 4	Student 6	Student 1	Student 6

$$\mathbf{M}_{ij} = \left\{ \begin{array}{l} \text{\# of times Student i has worked with Student j } \text{ if } i \neq j \\ 0 \text{ if } i = j \end{array} \right.$$

For example,

$\mathbf{M}_{23} = 2$ (# of times Student 2 has worked with Student 3)

$\mathbf{M}_{33} = 0$ because $i = j = 3$. This second part of the definition (when $i = j$) is referring to the diagonal elements. (The number 0 is chosen arbitrarily, choosing 3 makes just as much sense.)

Summer Teams		Fall Teams		Spring Teams	
Team 1	Team 2	Team 1	Team 2	Team 1	Team 2
Student 1	Student 4	Student 2	Student 5	Student 2	Student 3
Student 2	Student 5	Student 3	Student 1	Student 4	Student 5
Student 3	Student 6	Student 4	Student 6	Student 1	Student 6

$$\mathbf{M}_{ij} = \begin{cases} \text{\# of times Student i has worked with Student j} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

$$\mathbf{M} = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 & 3 & 0 \end{pmatrix}$$

We can think of this data as a social network of students.

$$\mathbf{M} = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 & 3 & 0 \end{pmatrix}$$

The matrix **M** is called the **adjacency** matrix for the network because it tells us which students are connected (adjacent). We can even draw a **graph** of this network, using 6 circles (*vertices*) to represent the students and connecting lines (*edges*) to represent their memberships on the same team.

Social Network Graph

$$\mathbf{M} = \begin{pmatrix} 0 & 2 & 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 2 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 1 & 3 & 0 \end{pmatrix}$$
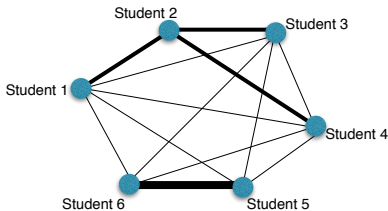
The thickness of the edge here represents how often the students have worked together, with thicker edges indicating partnerships that happened more often.

Describing Matrices and Vectors

- Size or Dimension of a matrix
- (i,j)-notation
- Notation
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Notation for Matrices, Vectors, and Scalars

In this course (and in most resources),

- Matrices will be named with **bold**, capital letters. For example,

$$\mathbf{M}, \quad \mathbf{P}, \quad \mathbf{A}, \quad \boldsymbol{\Sigma}$$

will always represent matrices.

- Vectors will be named with **bold**, lowercase letters. For example,

$$\mathbf{v}, \quad \mathbf{u}, \quad \mathbf{p}, \quad \mathbf{x}_2$$

will always represent vectors.

- Scalars will always be unbolded lowercase letters, often greek. For example:

$$\alpha, \quad \lambda, \quad c, \quad a_{32}, \quad v_2$$

will always represent scalars.

Notation for Matrices, Vectors, and Scalars

This notational convention helps us understand what we are looking at. For example, if we had an equation like

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \qquad \text{(and we will)}$$

We can immediately know what each part of the equation represents:

- $\mathbf{A}$ is a matrix
- $\mathbf{x}$ is a vector
- λ is a scalar

We don't know how to add or multiply these objects quite yet, but that's next!

For the following quantities, indicate whether the notation indicates a Matrix, Scalar, or Vector.

$$\mathbf{A} \qquad A_{ij} \qquad \mathbf{v} \qquad p_2$$

$$\lambda \qquad A_{22} \qquad \mathbf{p}_2 \qquad \mathbf{M}_{\star 2}$$

For the following quantities, indicate whether the notation indicates a Matrix, Scalar, or Vector.

$$\mathbf{A} - \text{matrix} \quad A_{ij} - \text{scalar} \quad \mathbf{v} - \text{vector} \quad p_2 - \text{scalar}$$

$$\sigma - \text{scalar} \quad A_{22} - \text{scalar} \quad \mathbf{p}_2 - \text{vector} \quad \mathbf{M}_{\star 2} - \text{vector}$$

Vector Geometry: n-space

- Recall "ordered pairs" or coordinates (x_1, x_2) live on the two-dimensional plane. In Linear Algebra, we call this plane "2-space."
- Our data points have more than 2 variables, say n.
- They are represented by *n***-tuples** which are nothing more than ordered lists of numbers:

$$(x_1, x_2, x_3, \ldots, x_n).$$

- An n-tuple defines a *vector* with the same n elements.
- "points" and "vectors" are interchangeable concepts.
- The difference is that a vector can be characterized by a **direction** and a magnitude (length).

Vector Geometry: n-space

- Recall that the symbol $\mathbb{R}$ denotes the scalar real numbers.
- These numbers have a direction on number line, either positive (right) or negative (left)!
- They also have a magnitude: $|x|$ tells us the distance between x and the origin.

Vector Geometry: n-space

- In Linear Algebra, we use the notation $\mathbb{R}^n$ to denote the set of all vectors with n elements.
- Thus, when we write $\mathbf{v} \in \mathbb{R}^5$ we are saying that $\mathbf{v}$ is a vector with 5 elements.
- Sometimes you'll see $\mathbf{A} \in \mathbb{R}^{6 \times 8}$ which means that $\mathbf{A}$ is a 6×8 matrix with real elements.

Points vs. Vectors

While the ideas of points and vectors are essentially interchangeable, it will help in certain applications to think about our data points in one context or the other.
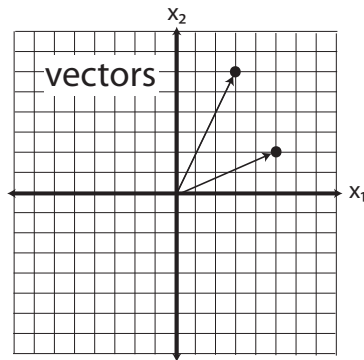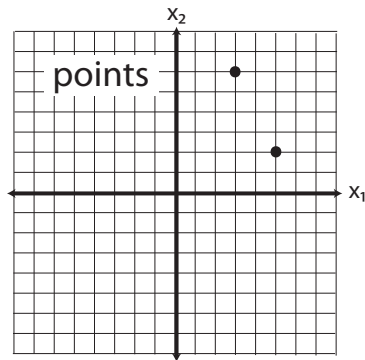
Describing Matrices and Vectors

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If $\mathbf{A}$ is $m \times n$ then $\mathbf{A}^T$ is the $n \times m$ matrix whose rows are the corresponding columns of $\mathbf{A}$.

For example, if $\mathbf{A}$ is a 3×4 matrix then $\mathbf{A}^T$ is a 4×3 matrix as follows:

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{pmatrix} \quad \mathbf{A}^T = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \\ A_{14} & A_{24} & A_{34} \end{pmatrix}$$

Just change the columns into rows! (or rows into columns, same thing!)

Let's do some more examples to make sure this is clear:

$$\mathbf{B} = \begin{pmatrix} 2 & -3 & -4 \\ 5 & -6 & -7 \\ -8 & 9 & 0 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} -1 & 2 \\ -3 & 6 \\ 7 & -9 \\ 5 & -1 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} 3 \\ -4 \\ 5 \\ 6 \end{pmatrix}$$

To find the transpose of these objects, we simply create new matrices by changing the rows into columns:

$$\mathbf{B}^T = \begin{pmatrix} 2 & 5 & -8 \\ -3 & -6 & 9 \\ -4 & -7 & 0 \end{pmatrix} \qquad \mathbf{M}^T = \begin{pmatrix} -1 & -3 & 7 & 5 \\ 2 & 6 & -9 & -1 \end{pmatrix}$$

$$\mathbf{x}^T = \begin{pmatrix} 3 & -4 & 5 & 6 \end{pmatrix}$$

Symmetric Matrix

A **symmetric** matrix is a matrix whose transpose is itself. For example,

$$\mathbf{B} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -6 & -7 \\ 1 & -7 & 0 \end{pmatrix} \qquad \mathbf{B}^T = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -6 & -7 \\ 1 & -7 & 0 \end{pmatrix}$$

is symmetric because $\mathbf{B}^T = \mathbf{B}$.

To be symmetric, a matrix must be square. Otherwise, the transpose wouldn't even have the same size as the original matrix!

To be symmetric, we must have that $\mathbf{B}_{ij} = \mathbf{B}_{ji}$ for all rows i and columns j. For example, $\mathbf{B}_{23} = \mathbf{B}_{32}$ above.

Ex: The Correlation Matrix

- When we have several variables to analyze, it's good practice to examine the pairwise correlations between variables.
- Suppose we have 4 variables, $x_1, x_2, x_3,$ and x_4.
- We use a *correlation matrix*, C, which is defined as follows:

$$C_{ij} = correlation(x_i, x_j).$$

Ex: The Correlation Matrix

Suppose our correlation matrix for our 4 variables is

$$\mathbf{C} = \begin{pmatrix} 1 & 0.3 & -0.9 & 0.1 \\ 0.3 & 1 & 0.8 & -0.5 \\ -0.9 & 0.8 & 1 & -0.6 \\ 0.1 & -0.5 & -0.6 & 1 \end{pmatrix}$$

- The diagonal elements of this matrix, C_{ii}, should always equal 1 because every variable is perfectly correlated with itself.
- In this example,

$$C_{13} = C_{31} = -0.9$$

indicates that $\mathbf{x}_1$ and $\mathbf{x}_3$ have a strong negative correlation.

The correlation matrix is always symmetric!

$C_{ij} = C_{ji}$ Because *correlation*(x_i, x_j) = *correlation*(x_j, x_i)

Given that,

$$\mathbf{A} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \\ 3 & -6 \end{pmatrix} \quad \mathbf{v}^T = \begin{pmatrix} 1 & 0 & -2 & 5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \\ B_{41} & B_{42} & B_{43} \end{pmatrix}$$

compute the following matrices:

$$\mathbf{A}^T = \qquad\qquad (\mathbf{A}^T)^T =$$

$$\mathbf{v} = \qquad\qquad \mathbf{B}^T =$$

Give an example of a 4×4 symmetric matrix:

Given that,

$$\mathbf{A} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \\ 3 & -6 \end{pmatrix} \quad \mathbf{v}^T = \begin{pmatrix} 1 & 0 & -2 & 5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \\ B_{41} & B_{42} & B_{43} \end{pmatrix}$$

compute the following matrices:

$$\mathbf{A}^T = \begin{pmatrix} 2 & -1 & 3 \\ -4 & 2 & -6 \end{pmatrix} \qquad (\mathbf{A}^T)^T = \begin{pmatrix} 2 & -4 \\ -1 & 2 \\ 3 & -6 \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 5 \end{pmatrix} \qquad \mathbf{B}^T = \begin{pmatrix} B_{11} & B_{21} & B_{31} & B_{41} \\ B_{12} & B_{22} & B_{32} & B_{42} \\ B_{13} & B_{23} & B_{33} & B_{43} \end{pmatrix}$$

Give an example of a 4×4 symmetric matrix, $\mathbf{S} =$
many possible, as long as $\mathbf{S} = \mathbf{S}^T$

Describing Matrices and Vectors

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The **identity matrix** is a square matrix with diagonal elements equal to 1 and all other elements equal to 0. The bold capital letter **I** is always reserved for the identity matrix. Sometimes a subscript is used to specify the dimensions of the matrix:

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The Elementary Vectors

The columns of the identity matrix are sometimes referred to as the **elementary vectors**. Elementary vectors have zeros everywhere except for a '1' in a single position. We write $\mathbf{e}_j$ to specify the jth column of the identity matrix:

$$\mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{e}_j = j^{th} row \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The vector $\mathbf{e}$ with no subscript is generally used to denote the vector of all ones (sometimes written as $\mathbf{1}$)

Diagonal Matrix

The identity matrix is a special case of a diagonal matrix $\mathbf{D}$, which is square and has

$$D_{ij} = 0 \quad \text{when } i \neq j$$

In other words, off-diagonal elements are equal to zero, for example,

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Since all other elements are zero, it is enough to specify the diagonal elements to create a diagonal matrix:

$$\mathbf{D} = diag\{2, -1, 3\} \quad \mathbf{S} = diag\{6, 3, 0, 2\}$$

Upper Triangular Matrix

An **upper triangular matrix** has zeros below the main diagonal:

$$\mathbf{M} = \begin{pmatrix} * & * & * & \ldots & * \\ 0 & * & * & \ldots & * \\ 0 & 0 & * & \ldots & * \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$$

(The asterisks represent any number - even potential 0's)

Lower Triangular Matrix

A **lower triangular matrix** has zeros above the main diagonal:

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ * & 0 & 0 & \ldots & 0 \\ * & * & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ * & * & * & * & 0 \end{pmatrix}$$

The Trace of a Matrix

The **Trace** of any square matrix **A**, written $tr(\mathbf{A})$ or $Trace(\mathbf{A})$ is the sum of it's diagonal elements:

$$tr(\mathbf{A}) = \sum_{i=1}^{n} A_{ii}$$

Let

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & 1 \\ 0 & 1 & -2 \\ -1 & \sqrt{2} & 3 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then,

$$tr(\mathbf{A}) = \sum_{i=1}^{3} A_{ii} = 3 + 1 + 3 = 7.$$

$$tr(\mathbf{D}) = \sum_{i=1}^{3} D_{ii} = 2 - 1 + 3 + 1 = 5$$

Write out the following matrices and then compute their Trace, if possible:

$$\mathbf{I}_5 \qquad \mathbf{D} = diag\{2, 6, 1\} \qquad \mathbf{e}_2 \in \mathbb{R}^4$$

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$$\mathbf{I}_5 \qquad \mathbf{D} = diag\{2, 6, 1\} \qquad \mathbf{e}_2 \in \mathbb{R}^4$$

$$\mathbf{I}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad tr(\mathbf{I}_5) = 5$$

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad tr(\mathbf{D}) = 9$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{trace only defined for square matrices}$$

Check your Understanding

The following are examples of triangular matrices. Are they upper or lower triangular?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The following are examples of triangular matrices. Are they upper or lower triangular?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

upper triangular   lower triangular   both upper and lower!