Chapter 6

Linear Independence

Linear Dependence/Independence

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is **linearly dependent** if we can express the zero vector, $\mathbf{0}$, as a *non-trivial* linear combination of the vectors.

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p = \mathbf{0}$$

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The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is **linearly independent** if the above equation has only the trivial solution, $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$.

Linear Dependence - Example

The vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix}$ are **linearly dependent** because

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#PerfectMulticollinearity!

Example - Determining Linear Independence

$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix}$ How can we tell if these vectors are linearly independent?

- Want to know if there are coefficients $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$$

- This creates a linear system!

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 2 & 3 & 7 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Just use Gauss-Jordan elimination to find out that

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

is one possible solution (there are free variables)!

Example - Determining Linear Independence

For a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2\mathbf{v}_3\}$,

- If the only solution was the trivial solution,

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

  Then we'd know that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

- $\Longrightarrow$ no free variables! Gauss-Jordan elimination on the vectors results in the identity matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Summary - Determining Linear Independence

The sum from our definition,

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p = \mathbf{0},$$

is simply a matrix-vector product

$$\mathbf{V}\boldsymbol{\alpha} = \mathbf{0}$$

where $\mathbf{V} = (\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \ldots \,|\, \mathbf{v}_p)$ and $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{pmatrix}$

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So all we need to do is determine whether the system of equations $\mathbf{V}\boldsymbol{\alpha} = \mathbf{0}$ has any non-trivial solutions.

Rank and Linear Independence

- If a set of vectors (think: *variables*) is not linearly independent, then the matrix that contains those vectors as columns (think: *data matrix*) is not full rank!
- The **rank** of a matrix can be defined as the number of linearly independent columns (or rows) in that matrix.
  - # of linearly independent rows = # of linearly independent columns
- In most data - # of rows > # of columns.
- So the maximum rank of a matrix is the # of columns - an $n \times m$ full rank matrix has $rank = m$.

Linear Independence

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Let $\mathbf{A}$ be an $n \times n$ matrix. The following statements are *equivalent*. (If one these statements is true, then all of these statements are true)

- $\mathbf{A}$ is invertible ($\mathbf{A}^{-1} exists$)
- $\mathbf{A}$ has full rank ($rank(\mathbf{A}) = n$)
- The columns of $\mathbf{A}$ are linearly independent
- The rows of $\mathbf{A}$ are linearly independent
- The system $\mathbf{Ax} = \mathbf{b}$ has a unique solution
- $\mathbf{Ax} = \mathbf{0} \Longrightarrow \mathbf{x} = \mathbf{0}$
- $\mathbf{A}$ is nonsingular
- $\mathbf{A} \xrightarrow{Gauss-Jordan} \mathbf{I}$

Check your understanding

Let $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

- Are the vectors $\mathbf{a}$ and $\mathbf{b}$ linearly independent?

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- What is the rank of the matrix $\mathbf{A} = (\mathbf{a}|\mathbf{b})$?
- Determine whether or not the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is a linear combination of the vectors **a** and **b**.

Check your understanding - Solution

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- What is the rank of the matrix $\mathbf{A} = (\mathbf{a}|\mathbf{b})$? Is **A** full rank?
  $rank(\mathbf{A}) = 2$ because there are two linearly independent columns. **A** is full rank.

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- Determine whether or not the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is a linear combination of the vectors **a** and **b**.

  Row reduce the augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 3 & 1 \\ 3 & 0 & 0 \\ 4 & 1 & 1 \end{array}\right)$$

  to find that the system is inconsistent. $\Longrightarrow$ No.

Why the fuss?

If our design matrix $\mathbf{X}$ is not full rank, then the matrix from the normal equations, $\mathbf{X}^T\mathbf{X}$ is also not full rank.

- $\mathbf{X}^T\mathbf{X}$ does not have an inverse.
- The normal equations do not have a unique solution!
- β's not uniquely determined.
- Infinitely many solutions.
- #PerfectMulticollinearity
- Breaks a fundamental assumption of MLR.

Example - Perfect vs. Severe Multicollinearity

Often times we'll run into a situation where variables are linearly independent, but only barely so. Take, for example, the following system of equations:

$$\beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2 = \mathbf{y}$$

where

$$\mathbf{x}_1 = \begin{pmatrix} 0.835 \\ 0.333 \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} 0.667 \\ 0.266 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 0.168 \\ 0.067 \end{pmatrix}$$

This system has an exact solution, $\beta_1 = 1$ and $\beta_2 = -1$.

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If we change this system only slightly, so that $\mathbf{y} = \begin{pmatrix} 0.168 \\ 0.066 \end{pmatrix}$ then the exact solution changes drastically to

$$\beta_1 = -666 \text{ and } \beta_2 = 834$$

.

The system is **unstable** because the columns of the matrix are so close to being linearly dependent!

Symptoms of Severe Multicollinearity

- Large fluctuations or flips in sign of the coefficients when a collinear variable is added into the model.
- Changes in significance when additional variables are added.
- Overall F-test shows significance when the individual t-tests show none.

These symptoms are bad enough on their own, but the real consequence of this type of behavior is that seen in the previous example. A very small change in the underlying system of equations (like a minuscule change in a target value y_i) can produce dramatic changes to our parameter estimates!