Chapter 4

Solving Systems of Equations

3 Scenarios for Solutions

There are three general situations we may find ourselves in when attempting to solve systems of equations:

1. The system could have one unique solution.
2. The system could have infinitely many solutions (sometimes called *underdetermined*).
3. The system could have no solutions (sometimes called *overdetermined* or *inconsistent*).

The method of solving these systems is the same in each scenario.

Gaussian Elimination

- Suppose we have the following simple system of equations:

$$\begin{cases} x_1 + 2x_2 = 11 \\ x_1 + \quad x_2 = 6 \end{cases}$$

- One way to solve this system of equations is to subtract the second equation from the first.
- Perform subtraction on the left hand and right hand sides of the equation:

$$\left(\begin{array}{cccc} & x_1 & + & 2x_2 \\ - & (x_1 & + & x_2) \\ \hline & & & x_2 \end{array} \right) = \left(\begin{array}{c} 11 \\ -6 \\ \hline 5 \end{array} \right)$$

- Left with one much simpler equation,

$$x_2 = 5$$

Gaussian Elimination

- Suppose we have the following simple system of equations:

$$\begin{cases} x_1 + 2x_2 = 11 \\ x_1 + x_2 = 6 \end{cases}$$

- Pair this simpler equation,

$$x_2 = 5$$

  With one of the original equations,

$$x_1 + 2x_2 = 11$$

  and we have a system whose solution becomes clear through substitution.

$$x_1 = 1, x_2 = 5$$

- This final process of substitution is often called **back substitution.**

Row Operations

For any system of equations, there are 3 operations which will not change the solution set. Taking our simple system from the previous example, we'll examine these three operations concretely...

1) Interchanging Rows

$$\begin{cases} x_1 + 2x_2 = 11 \\ x_1 + x_2 = 6 \end{cases}$$

- Interchanging the order of the equations.

  Clearly,

$$\begin{cases} x_1 + 2x_2 = 11 \\ x_1 + x_2 = 6 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 = 6 \\ x_1 + 2x_2 = 11 \end{cases}$$

2) Multiplying one row by a scalar constant

$$\begin{cases} x_1 + 2x_2 = 11 \\ x_1 + x_2 = 6 \end{cases}$$

- Multiplying both sides of one equation by a constant.

  The second equation doesn't really change if multiplied by 2,

$$\begin{cases} x_1 + 2x_2 = 11 \\ x_1 + x_2 = 6 \end{cases} \Leftrightarrow \begin{cases} x_1 + 2x_2 = 11 \\ 2x_1 + 2x_2 = 12 \end{cases}$$

3)Replacing one equation with a combination

$$\begin{cases} x_1 + 2x_2 = 11 \\ x_1 + x_2 = 6 \end{cases}$$

- Replace one equation by a combination of itself plus a multiple of another equation.

  As was demonstrated previously, we can replace equation 2 with the combination of (equation 2 - equation 1).

$$\begin{cases} x_1 + 2x_2 = 11 \\ x_1 + x_2 = 6 \end{cases} \Leftrightarrow \begin{cases} x_1 + 2x_2 = 11 \\ x_2 = 5 \end{cases}$$

Triangular Systems

Using these row operations, we can transform any system of equations into one that is *triangular*. A **triangular system** is one that can be solved by back substitution, for example:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 14 \\ \qquad\quad x_2 + x_3 = 6 \\ \qquad\qquad\qquad x_3 = 1 \end{cases}$$

Example - Transforming a System to Triangular

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 - 2x_2 + 2x_3 = 4 \\ x_1 + 2x_2 - x_3 = 2 \end{cases}$$

- We will want to eliminate the variable x_1 from two of the equations:
  a. Replace equation 2 with (equation 2 - equation 1).
  b. Replace equation 3 with (equation 3 - equation 1).
- Then, our system becomes:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ - 3x_2 + x_3 = 3 \\ x_2 - 2x_3 = 1 \end{cases}$$

Example - Transforming a System to Triangular

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ - 3x_2 + x_3 = 3 \\ x_2 - 2x_3 = 1 \end{cases}$$

- Next, we will want to eliminate the variable x_2 from the third equation. We can do this by replacing equation 3 with (equation 3 + $\frac{1}{3}$ equation 2) OR...
- Swap equations 2 and 3 first (to avoid fractions).

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_2 - 2x_3 = 1 \\ - 3x_2 + x_3 = 3 \end{cases}$$

Example - Transforming a System to Triangular

$$\begin{cases} x_1 + \ \ x_2 + \ \ x_3 = 1 \\ \qquad\quad x_2 - 2x_3 = 1 \\ \quad\ - 3x_2 + \ \ x_3 = 3 \end{cases}$$

- NOW eliminate x_2 from the third equation
- Replace equation 3 with (equation 3 + 3*equation 2).

$$\begin{cases} x_1 + x_2 + \ \ x_3 = 1 \\ \qquad\ x_2 - 2x_3 = 1 \\ \qquad\quad\ - 5x_3 = 6 \end{cases}$$

Example - Transforming a System to Triangular

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ \phantom{x_1 + {}} x_2 - 2x_3 = 1 \\ - 5x_3 = 6 \end{cases}$$

Now that our system is in triangular form, we can use substitution to solve for all of the variables:

$$x_1 = 3.6 \quad x_2 = -1.4 \quad x_3 = -1.2$$

This is the procedure for Gaussian Elimination, which we will now formalize in its matrix version.

The Augmented Matrix

- The **augmented matrix** contains all of the numerical information from our system of equations.
- Matrix that contains all of the coefficients of the equations, augmented with an extra column containing the right hand sides of the equations.

The Augmented Matrix

If our system is:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Then the corresponding augmented matrix is

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{12} & a_{33} & b_3 \end{array}\right)$$

Row Operations on The Augmented Matrix

- The augmented matrix contains all of the information needed to perform the three operations outlined previously.
- We will formalize these operations as they pertain to the rows of the augmented matrix.

Row Operations for Gaussian Elimination

Gaussian Elimination is performed on an augmented matrix by using the three **elementary row operations**:

1. Swap rows i and j.
2. Replace row i by a nonzero multiple of itself.
3. Replace row i by a linear combination of itself plus a multiple of row j.

Row Operations for Gaussian Elimination

The ultimate goal of Gaussian elimination is to transform an augmented matrix $\mathbf{A}$ into an **upper-triangular matrix** which allows for solving via back substitution.

$$\mathbf{A} \to \left(\begin{array}{cccc|c} t_{11} & t_{12} & \dots & t_{1n} & c_1 \\ 0 & t_{22} & \dots & t_{2n} & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & t_{nn} & c_n \end{array}\right)$$

Pivots

The key to this process at each step is to focus on one position, called the *pivot position* or simply *pivot*, and try to eliminate all terms below this position using the three row operations.

$$\begin{aligned} \boxed{x_1} + x_2 + x_3 &= 1 \\ x_1 - 2x_2 + 2x_3 &= 4 \\ x_1 + 2x_2 - x_3 &= 2 \end{aligned}$$

$$\Longrightarrow$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right)$$

Example - Row Operations on the Augmented Matrix

After we identify the current pivot, our goal is to eliminate the numbers below (circled) using the row with the pivot.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right) \xrightarrow{CurrentPivot} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ \textcircled{1} & -2 & 2 & 4 \\ \textcircled{1} & 2 & -1 & 2 \end{array}\right)$$

For instance, we'd replace row 2 by the combination

$$(row\,2 - row\,1).$$

Our shorthand notation for this will be R2′ = R2-R1.

Example - Row Operations on the Augmented Matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right) \xrightarrow{R2'=R2-R1} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ \color{red}{0} & \color{red}{-3} & \color{red}{1} & \color{red}{3} \\ \textcircled{1} & 2 & -1 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & -3 & 1 & 3 \\ \textcircled{1} & 2 & -1 & 2 \end{array}\right) \xrightarrow{R3'=R3-R1} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & -3 & 1 & 3 \\ \color{red}{0} & \color{red}{1} & \color{red}{-2} & \color{red}{1} \end{array}\right)$$

Example - Row Operations on the Augmented Matrix

Now that we have eliminated each of the circled elements below the current pivot, we will continue on to the next pivot, which is -3.

$$\xrightarrow{NextPivot} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{-3} & 1 & 3 \\ 0 & \textcircled{1} & -2 & 1 \end{array}\right)$$

Looking into the future, we can either do the operation $R3' = R3 + \frac{1}{3}R2$ or we can interchange rows 2 and 3 to avoid fractions in our next calculation. (*note: either way you proceed will lead you to the same solution!*)

Example - Row Operations on the Augmented Matrix

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{-3} & 1 & 3 \\ 0 & \textcircled{1} & -2 & 1 \end{array}\right) \xrightarrow{R2 \leftrightarrow R3} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & -2 & 1 \\ 0 & \textcircled{-3} & 1 & 3 \end{array}\right)$$

Now that the current pivot is equal to 1, we can more easily eliminate the circled entry below it using combinations with the pivot row.

$$\xrightarrow{R3' = R3 + 3R2} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & \boxed{-5} & 6 \end{array}\right)$$

Example - Row Operations on the Augmented Matrix

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & \boxed{-5} & 6 \end{array}\right)$$

- When all the pivots have been reached, the augmented matrix is said to be in **row-echelon form**.
- This simply means that all of the entries below the pivots are equal to 0.
- Left part of **A** is upper-triangular.

Example - Row Operations on the Augmented Matrix

The augmented matrix can be transformed back into equation form now that it is in a triangular form:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ \phantom{x_1 + {}} x_2 - 2x_3 = 1 \\ \phantom{x_1 + x_2 - {}} 5x_3 = -6 \end{cases}$$

Which can (again) be solved by back substitution to get:

$$x_1 = 3.6 \quad x_2 = -1.4 \quad x_3 = -1.2$$

Gaussian Elimination Summary: Step-by-step

1. Identify the first pivot element. The first pivot element should be located in the first row (if this entry is zero, we must interchange rows so that it is non-zero).
2. Eliminate all elements below the pivot using the combination row operation.
3. Determine the next pivot and go back to step 2.
   - Only nonzero numbers are allowed to be pivots! If a coefficient in a pivot position is ever 0, then the rows of the matrix should be interchanged to find a nonzero pivot. If this is not possible then we continue on to the next possible column where a pivot position can be created.
4. When the entries below all of the pivots are equal to zero, the process stops. The augmented matrix is said to be in *row-echelon form*
   - *Triangular* system of equations
   - Solve using back substitution

Gaussian Elimination and Rank of a Matrix

The **rank** of a matrix can be defined as the number of pivot elements used in Gaussian Elimination.

- **full row-rank** if there is a pivot in every row.
- **full column-rank** if there is a pivot in every column.
- **full rank** if it is either full row-rank or full-column rank.

A square matrix with full rank has an inverse!

Check your Understanding

Use Gaussian Elimination to solve the following system of equations:

$$\begin{cases} \qquad\quad 2x_2 + 3x_3 = 8 \\ 2x_1 + 3x_2 + 1x_3 = 5 \\ \ x_1 - \ \ x_2 - 2x_3 = -5 \end{cases}$$

Is the coefficient matrix full rank? How do you know?

Check your Understanding - Solution

Use Gaussian Elimination to solve the following system of equations:

$$\begin{cases} \quad\quad\quad 2x_2 + 3x_3 = 8 \\ 2x_1 + 3x_2 + 1x_3 = 5 \\ \ x_1 - \ x_2 - 2x_3 = -5 \end{cases}$$

The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Is the coefficient matrix full rank? How do you know? Yes, the coefficient matrix is full rank. It has a pivot element in every row. (Similarly you could say it has a pivot element in every column)

Gauss-Jordan Elimination

In Gauss-Jordan elimination, we do not stop when the augmented matrix is in row-echelon form. Instead, two additional things are required:

1. Force all the pivot elements to equal 1
2. Eliminate entries *above* the pivot elements to reach what's called **reduced row echelon form**.

Example: Gauss-Jordan Elimination

We begin with a system of equations, and transform it into an augmented matrix:

$$\begin{cases} \quad\quad\quad x_2 - x_3 = 3 \\ -2x_1 + 4x_2 - x_3 = 1 \\ -2x_1 + 5x_2 - 4x_3 = -2 \end{cases} \Longrightarrow \left(\begin{array}{ccc|c} 0 & 1 & -1 & 3 \\ -2 & 4 & -1 & 1 \\ -2 & 5 & -4 & -2 \end{array}\right)$$

We start by locating our first pivot element. This element cannot be zero, so we will have to swap rows to bring a non-zero element to the pivot position.

Example: Gauss-Jordan Elimination

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 3 \\ -2 & 4 & -1 & 1 \\ -2 & 5 & -4 & -2 \end{array}\right) \xrightarrow{R1 \leftrightarrow R2} \left(\begin{array}{ccc|c} \boxed{\text{-2}} & 4 & -1 & 1 \\ 0 & 1 & -1 & 3 \\ -2 & 5 & -4 & -2 \end{array}\right)$$

Now that we have a non-zero pivot, we will want to do two things:

- Use the pivot to eliminate all of the elements below it (as with Gaussian elimination)
- Make the pivot element equal to 1.

It does not matter what order we perform these two tasks in. Here, we will have an easy time eliminating using the -2 pivot.

Example: Gauss-Jordan Elimination

$$\left(\begin{array}{ccc|c} \boxed{-2} & 4 & -1 & 1 \\ 0 & 1 & -1 & 3 \\ -2 & 5 & -4 & -2 \end{array}\right) \xrightarrow{R3'=R3-R1} \left(\begin{array}{ccc|c} \boxed{-2} & 4 & -1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & -3 & -3 \end{array}\right)$$

Now, as promised, we will make our pivot equal to 1.

$$\left(\begin{array}{ccc|c} \boxed{-2} & 4 & -1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & -3 & -3 \end{array}\right) \xrightarrow{R1'=-\frac{1}{2}R1} \left(\begin{array}{ccc|c} \boxed{1} & -2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 & 3 \\ 0 & 1 & -3 & -3 \end{array}\right)$$

Example: Gauss-Jordan Elimination

We have finished our work with this pivot, and now we move on to the next one. Since it is already equal to 1, the only thing left to do is use it to eliminate the entries below it:

$$\left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 1 & -3 & -3 \end{array}\right) \xrightarrow{R3'=R3-R2} \left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \boxed{1} & -1 & 3 \\ \color{red}{0} & \color{red}{0} & \color{red}{-2} & \color{red}{-6} \end{array}\right)$$

Example: Gauss-Jordan Elimination

And then we move onto our last pivot. This pivot has no entries below it to eliminate, so all we must do is turn it into a 1:

$$\left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & \frac{-1}{2} \\ 0 & 1 & -1 & 3 \\ 0 & 0 & \boxed{\text{-2}} & -6 \end{array}\right) \xrightarrow{R3' = -\frac{1}{2}R3} \left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 & 3 \\ \color{red}{0} & \color{red}{0} & \color{red}{\boxed{1}} & \color{red}{3} \end{array}\right)$$

Example: Gauss-Jordan Elimination

Now, what really differentiates Gauss-Jordan elimination from Gaussian elimination is the next few steps. Here, our goal will be to use the pivots to eliminate all of the entries *above* them.

Example: Gauss-Jordan Elimination

We'll start at the southeast corner on the current pivot. We will use that pivot to eliminate the elements above it:

$$\left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 & 3 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right) \xrightarrow{R2'=R2+R3} \left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & -\frac{1}{2} \\ \color{red}{0} & \color{red}{1} & \color{red}{0} & \color{red}{6} \\ 0 & 0 & \boxed{1} & 3 \end{array}\right)$$

Example: Gauss-Jordan Elimination

$$\left(\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 6 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right) \xrightarrow{R1'=R1-\frac{1}{2}R3} \left(\begin{array}{ccc|c} \color{red}{1} & \color{red}{-2} & \color{red}{0} & \color{red}{-2} \\ 0 & 1 & 0 & 6 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right)$$

We're almost done! One more pivot with elements above it to be eliminated

Example: Gauss-Jordan Elimination

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ 0 & \boxed{1} & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow{R1'=R1+2R2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & \boxed{1} & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

And we've reached what's called **reduced row echelon form**.

Example: Gauss-Jordan Elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

How does this help us? Well, let's transform back to a system of equations:

$$\begin{cases} x_1 \qquad\qquad = 10 \\ \qquad x_2 \qquad = 6 \\ \qquad\qquad x_3 = 3 \end{cases}$$

The solution is simply what's left in the right hand column of the augmented matrix.

Gauss-Jordan Elimination Summary

1. Identify the first pivot element. The first pivot element should be located in the first row (if this entry is zero, we must interchange rows so that it is non-zero).
2. The pivot element should be equal to 1. If it is not, we simply multiply the row by a constant to make it equal 1 (or interchange rows, if possible).
3. Eliminate (zero-out) all elements below the pivot using the combination row operation.
4. Determine the next pivot and go back to step 2.
5. When the last pivot is equal to 1, begin to eliminate all the entries above the pivot positions.
6. When all entries above and below each pivot element are equal to zero, the augmented matrix is said to be in *reduced row echelon form* and the Gauss-Jordan elimination process is complete.

Check your Understanding

Use Gauss-Jordan Elimination to solve the following system of equations:

$$\begin{cases} \phantom{2x_1 + {}} 2x_2 + 3x_3 = 8 \\ 2x_1 + 3x_2 + 1x_3 = 5 \\ x_1 - x_2 - 2x_3 = -5 \end{cases}$$

Check your Understanding - Solution

Use Gauss-Jordan Elimination to solve the following system of equations:

$$\begin{cases} \quad\quad\;\; 2x_2 + 3x_3 = 8 \\ 2x_1 + 3x_2 + 1x_3 = 5 \\ \;\; x_1 - \;\; x_2 - 2x_3 = -5 \end{cases}$$

The solution is still

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Three Types of Systems

There are 3 situations that may arise when solving a system of equations:

Case 1: The system could have one unique solution (this is the case with the examples thus far).

Case 2: The system could have no solutions (sometimes called *overdetermined* or *inconsistent*).

Case 3: The system could have infinitely many solutions (sometimes called *underdetermined*).

The Unique Solution Case

In the previous examples, we have seen this situation where the system of equations leads us to one precise solution.

The Inconsistent Case

The second case scenario is a very specific one. For a system of equations to be **inconsistent** and have no solutions, at least one equation reduces to the form

$$0 = \alpha$$

where α is nonzero after Gaussian Elimination:

$$\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & \alpha \end{array}\right)$$

The third row indicates that

$$0x_1 + 0x_2 + 0x_3 = \alpha$$

where $\alpha \neq 0$, which is a contradiction.

Example - Identifying an Inconsistent System

$$\begin{cases} x - y + z = 1 \\ x - y - z = 2 \\ x + y - z = 3 \\ x + y + z = 4 \end{cases}$$

Using the augmented matrix and Gaussian elimination, we take the following steps:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 2 \\ 1 & 1 & -1 & 3 \\ 1 & 1 & 1 & 4 \end{array}\right) \xrightarrow[]{\substack{R2'=R2-R1 \\ R3'=R3-R1 \\ R4'=R4-R1}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 2 & 0 & 3 \end{array}\right)$$

Example - Identifying an Inconsistent System

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 2 & 0 & 3 \end{array}\right) \xrightarrow{R4 \leftrightarrow R2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & -2 & 1 \end{array}\right) \xrightarrow{R3' = R3 - R2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R4' = R4 - R3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

In this final step, we see our contradiction equation, $0 = 2$. Since this is obviously impossible, we conclude that the system is inconsistent.

Example - Identifying an Inconsistent System

- Sometimes inconsistent systems are referred to as *over-determined*.
- Usually because more equations than variables.
- Holding too many demands for a small set of variables!
- This is precisely the situation in which we find ourselves when we approach linear regression!
- Since we can't find an exact solution, we have to try to get the left and right hand sides as close as possible.
- This is done using the Least Squares method (coming soon...)

The Infinite Solutions Case

Consider the following system of equations written as an augmented matrix, and its reduced row echelon form after Gauss-Jordan elimination.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow{Gauss-Jordan} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Notice the following:

- The reduced form has a row that is completely 0.
  - One of the equations was completely eliminated using a combination of the others.
  - It contained unnecessary/redundant information.
- There are only 2 pivot elements.
  - Last entries in third column could not be eliminated.
  - Characteristic of what's called a **free-variable**.

The Infinite Solutions Case

Translating our reduced system back to equations:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{cases} x_1 \quad\;\; + x_3 = 0 \\ \quad\;\; x_2 + x_3 = 0 \end{cases}$$

Answer depends on the variable x_3, which is *free* to take on any value. Suppose that $x_3 = \alpha$. Then our solution would be:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\alpha \\ -\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Any scalar multiple of the vector $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ is a solution! There are infinitely many solutions!

The Infinite Solutions Case

A system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ has infinitely many solutions if the system is consistent and *any* of the following conditions hold:

1. After Gauss-Jordan elimination, at least one row of the matrix has every element equal to 0.
2. The number of variables is greater than the number of equations.
3. One of the equations is a linear combination of the others.
4. There is at least one *free variable* presented in the reduced row echelon form.
5. The number of pivots is less than the number of variables.

Example - The Infinite Solutions Case

For this reduced system, characterize the set of all solutions.

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 \quad\; + x_3 + 2x_4 = 0 \\ \quad x_2 + x_3 - \;\; x_4 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 \quad = -x_3 - 2x_4 \\ \quad x_2 = -x_3 + x_4 \end{cases}$$

Now we have *two* variables which are free to take on any value. Thus, let

$$x_3 = s \quad \text{and} \quad x_4 = t$$

Then, our solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s - 2t \\ -s + t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Example - The Infinite Solutions Case

Then, our solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s-2t \\ -s+t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

so any linear combination of the vectors

$$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

will provide a solution to this system.

Solving Matrix Equations

What happens when we have a matrix equation like

$$\mathbf{AX} = \mathbf{B}?$$

This situation is an easy extension of our previous problem because we are essentially solving the same system of equation with several different right-hand-side vectors (the columns of **B**).

Let's look at a 2×2 example to get a feel for this. We'll dissect the following matrix equation into two different systems of equations:

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 4 & 5 \end{pmatrix}.$$

Solving Matrix Equations

This matrix equation represents 4 separate equations which we'll combine into two systems:

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

The augmented matrices for these two systems:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 4 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 5 \end{array}\right)$$

Solving Matrix Equations

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 4 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 5 \end{array}\right)$$

- When performing Gauss-Jordan elimination on these two augmented matrices, how are the row operations going to differ?
  - They're not!
- The same row operations will be used for each augmented matrix - the only thing that will differ is how these row operations will affect the right hand side vectors.

Thus, it is possible for us to keep track of those differences in one larger augmented matrix :

$$\left(\begin{array}{cc|cc} 1 & 1 & 3 & 3 \\ 2 & 1 & 4 & 5 \end{array}\right)$$

We can then perform the row operations on both right-hand sides at once.

Solving Matrix Equations

$$\left(\begin{array}{cc|cc} 1 & 1 & 3 & 3 \\ 2 & 1 & 4 & 5 \end{array}\right) \xrightarrow{R2'=R2-2R1} \left(\begin{array}{cc|cc} 1 & 1 & 3 & 3 \\ 0 & -1 & -2 & -1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 3 & 3 \\ 0 & -1 & -2 & -1 \end{array}\right) \xrightarrow{R2'=-1R2} \left(\begin{array}{cc|cc} 1 & 1 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right) \xrightarrow{R1'=R1-R2} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

Now again, remembering the situation from which we came, we have the equivalent system:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

So we can conclude that

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

and we have solved our system.
This method is particularly useful when finding the inverse of a matrix.

Solving for the Inverse of a Matrix

For any square matrix $\mathbf{A}$, we know the inverse matrix ($\mathbf{A}^{-1}$), if it exists, satisfies the following matrix equation,

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}.$$

Thus, using the Gauss-Jordan method with multiple right hand sides, we can solve for the inverse of any matrix.

$$\left(\mathbf{A} \mid \mathbf{I}\right) \xrightarrow{Gauss-Jordan} \left(\mathbf{I} \mid \mathbf{A}^{-1}\right)$$

If this is possible then the matrix on the right is the inverse of $\mathbf{A}$. If this is not possible then $\mathbf{A}$ does not have an inverse.

Example - Solving for the Inverse of a Matrix

To find the inverse of

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & -1 \\ 0 & -1 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

using Gauss-Jordan Elimination, we first set up the augmented matrix as $(\mathbf{A} \mid \mathbf{I})$:

$$\left(\begin{array}{ccc|ccc} -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Example - Solving for the Inverse of a Matrix

We then proceed with Gauss-Jordan Elimination to transform the left hand side into the identity matrix:

$$\left(\begin{array}{ccc|ccc} -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R3'=R3+2R1} \left(\begin{array}{ccc|ccc} -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 3 & -2 & 2 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 3 & -2 & 2 & 0 & 1 \end{array}\right) \xrightarrow[R3'=R3+3R2]{R1'=-1R1} \left(\begin{array}{ccc|ccc} 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 \end{array}\right)$$

Example - Solving for the Inverse of a Matrix

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 \end{array}\right) \xrightarrow[R2'=R2-R3]{R1'=R1-R3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -3 & -3 & -1 \\ 0 & -1 & 0 & -2 & -2 & -1 \\ 0 & 0 & 1 & 2 & 3 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -3 & -3 & -1 \\ 0 & -1 & 0 & -2 & -2 & -1 \\ 0 & 0 & 1 & 2 & 3 & 1 \end{array}\right) \xrightarrow[R1'=R1+2R2]{R2'=-1R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 & 3 & 1 \end{array}\right)$$

Finally, we have completed our task. The inverse of $\mathbf{A}$ is the matrix on the right hand side of the augmented matrix!

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

Check your Understanding

Use the same method to determine the inverse of

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

Check your Understanding - Solution

Use the same method to determine the inverse of

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

The previous example actually tells us the answer we're looking for. If **B** is the inverse of **A** then **A** is also the inverse of **B**! So you should reach the solution

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & -1 \\ 0 & -1 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

Example - Inverse of a Diagonal Matrix

A full rank diagonal matrix (one with no zero diagonal elements) has a particularly neat and tidy inverse.

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \sqrt{5} \end{pmatrix}$$

Simple Gauss-Jordan Elimination:

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 & 1 \end{array}\right) \xrightarrow[]{\substack{R1'=\frac{1}{3}R1 \\ R2'=-\frac{1}{2}R2 \\ R3'=\frac{1}{\sqrt{5}}R3}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{\sqrt{5}} \end{array}\right)$$

Thus, the inverse of **D** is:

$$\mathbf{D}^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$