Worksheet - Lecture 14

Partitioned Matrix Arithmetic

1. Suppose I want to compute the matrix product $\mathbf{A} = \mathbf{UDV}^T$ where $\mathbf{U}$ is $n \times r$, $\mathbf{D}$ is an $r \times r$ diagonal matrix, $\mathbf{D} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$, and $\mathbf{V}^T$ is $r \times p$. *(Side note: we will quite often want to compute such a matrix product – this is the form of the singular value decomposition (SVD)! The following exercise is not just for fun - what you end up with in part b is exactly how we will want to write the SVD to best understand how it works.)*

   a. Using what you know about multiplication by diagonal matrices, if we view the matrix $\mathbf{U}$ as a collection of columns,

   $$\mathbf{U} = (\mathbf{U}_1|\mathbf{U}_2|\mathbf{U}_3|\ldots|\mathbf{U}_r)$$

   then how would I write the same partition of the matrix $\mathbf{UD}$?

   $$\mathbf{UD} = (?|?|?|\ldots|?)$$

   Diagonal Matrix on Right simply Scales the columns of U by diagonal elements

   $$UD = (\sigma_1 U_1 | \sigma_2 U_2 | \sigma_3 U_3 | \ldots | \sigma_r U_r)$$

   Keep in mind that when multiplying matrices/vectors by scalars, it is always preferable to write the scalar first ($\sigma\mathbf{x}$ rather than $\mathbf{x}\sigma$)

   b. Now, using the above representation for $\mathbf{UD}$, what happens when I multiply by the matrix $\mathbf{V}^T$, viewed as a collection of rows,

   $$\mathbf{V}^T = \begin{pmatrix} \mathbf{V}_1^T \\ \hline \mathbf{V}_2^T \\ \hline \mathbf{V}_3^T \\ \hline \vdots \\ \hline \mathbf{V}_r^T \end{pmatrix} ?$$

   $$\mathbf{UDV}^T = ?$$

   $$(\sigma_1 U_1 | \sigma_2 U_2 | \ldots | \sigma_r U_r) \begin{pmatrix} V_1^T \\ \hline V_2^T \\ \hline \vdots \\ \hline V_r^T \end{pmatrix} = \boxed{\sigma_1 U_1 V_1^T + \sigma_2 U_2 V_2^T + \ldots + \sigma_r U_r V_r^T}$$

   (Hint: your answer should be a sum. Each term in the sum should be an outer product.)

2. Consider

$$\mathbf{A}=\begin{pmatrix}-1 & 2 & 4 & 1 & 0\\ 1 & 0 & -1 & -2 & 1\\ 2 & -1 & 3 & 1 & 2\\ 1 & 2 & 3 & 4 & 3\\ -1 & -2 & 0 & 1 & 2\end{pmatrix} \quad \text{and} \quad \mathbf{x}=\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}$$

Partition these into submatrices (regions/blocks) conformably for multiplication as follows:

$$\mathbf{A}=\left(\begin{array}{c|c|c}\mathbf{A}_{00} & \mathbf{a}_{01} & \mathbf{A}_{02}\\ \hline \mathbf{a}_{10}^T & \alpha_{11} & \mathbf{a}_{12}^T\\ \hline \mathbf{A}_{20} & \mathbf{a}_{21} & \mathbf{A}_{22}\end{array}\right) \qquad \mathbf{x}=\left(\begin{array}{c}x_0\\ \hline \chi_1\\ \hline x_2\end{array}\right)$$

Where $\mathbf{A}_{00}$ is a 3×3 matrix, $\mathbf{x}_0 \in \mathbb{R}^3$, α_{11} is a scalar and χ_1 is a scalar. Show with lines how $\mathbf{A}$ and $\mathbf{x}$ are partitioned below:

$$\mathbf{A}=\left(\begin{array}{ccc|c|c}-1 & 2 & 4 & 1 & 0\\ 1 & 0 & -1 & -2 & 1\\ 2 & -1 & 3 & 1 & 2\\ \hline 1 & 2 & 3 & 4 & 3\\ \hline -1 & -2 & 0 & 1 & 2\end{array}\right) \quad \text{and} \quad \mathbf{x}=\left(\begin{array}{c}1\\2\\3\\ \hline 4\\ \hline 5\end{array}\right)$$

3. For all matrices $\mathbf{A}_{n\times k}$ and $\mathbf{B}_{k\times n}$, show that the block matrix

$$\mathbf{L}=\begin{pmatrix}\mathbf{I}-\mathbf{BA} & \mathbf{B}\\ 2\mathbf{A}-\mathbf{ABA} & \mathbf{AB}-\mathbf{I}\end{pmatrix}\begin{pmatrix}I-BA & B\\ 2A-ABA & AB-I\end{pmatrix}$$

satisfies the property $\mathbf{L}^2=\mathbf{I}$. *Hint: Perform block matrix multiplication for each of the four separate blocks in the result, simplifying each expression as much as possible.*

(1,1) - Block: $(I-BA)(I-BA) + B(2A-ABA)$
$= I - \cancel{2BA} + \cancel{BABA} + \cancel{2BA} - \cancel{BABA} = I$

(1,2) - Block: $(I-BA)B + B(AB-I)$
$= B - BAB + BAB - B = 0$

(2,1) - Block: $(2A-ABA)(I-BA) + (AB-I)(2A-ABA)$
$= \cancel{2A} - \cancel{ABA} - \cancel{2ABA} + \cancel{ABABA} + \cancel{2ABA} - \cancel{ABABA} - \cancel{2A} + \cancel{ABA} = 0$

(2,2) - Block: $(2A-ABA)B + (AB-I)(AB-I)$
$= \cancel{2AB} - \cancel{ABAB} + \cancel{ABAB} - \cancel{2AB} + I = I$

So, Final result $\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix} = I$