Worksheet - Lecture 15

Norms and Distance Measures

1. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -4 \\ -2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

   a. Determine the Euclidean distance between $\mathbf{u}$ and $\mathbf{v}$.

   $$\|u-v\|_2 = \sqrt{(1-1)^2 + (2+1)^2 + (-4-1)^2 + (-2+1)^2} = \sqrt{35}$$

   b. Find a vector of unit length in the direction of $\mathbf{u}$.

   $\|u\|_2 = 5$ So $\frac{1}{5} \cdot u$ is a unit vector : $\begin{pmatrix} 1/5 \\ 2/5 \\ -4/5 \\ -2/5 \end{pmatrix}$

   c. Find the 1- and ∞-norms of $\mathbf{u}$ and $\mathbf{v}$.

   $\|u\|_1 = 9$ $\quad \|v\|_1 = 4$

   $\|u\|_\infty = 4$ $\quad \|v\|_\infty = 1$

   d. Find the Manhattan distance between $\mathbf{u}$ and $\mathbf{v}$.

   $$\|u-v\|_1 = |1-1| + |2+1| + |-4-1| + |-2+1| = 9$$

2. What is the 2-norm of a unit vector?

   1! That's the definition

3. Describe in words what happens when you take the inner product of a vector with itself, $\mathbf{x}^T\mathbf{x}$. How does this computation relate to the 2-norm of $\mathbf{x}$, $\|\mathbf{x}\|_2$?

   You get the sum of squared elements, or $\|x\|_2^2$.
   In other words, $\|x\|_2 = \sqrt{x^T x}$

4. (*True/False*) When we have a system of equations for regression analysis,

$$\mathbf{X}\boldsymbol{\beta} = \mathbf{y}$$

which has no exact solutions, the goal of the Least-Squares method is to find a solution $\hat{\boldsymbol{\beta}}$ such that

$$\|\mathbf{X}\hat{\boldsymbol{\beta}} - \mathbf{y}\|_2^2$$ = sum of squared error

is minimized.

(In case the notation looks confusing, that is the two-norm squared, $\|\mathbf{x}\|_2^2 = (\|\mathbf{x}\|_2)^2$.)

$\boxed{\text{TRUE}}$