Worksheet - Lecture 4

Matrix Arithmetic Part Two

1. Use the following matrices to answer the questions:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 8 \\ 3 & 0 & -2 \\ 8 & -2 & -3 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 1 & 8 & -2 & 5 \\ 2 & 8 & 1 & 7 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\mathbf{H} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \quad \mathbf{W} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix}$$

a. Circle the matrix products that are possible and specify their resulting dimensions:

AM 3x3 2x4

$\mathbf{W}^T\mathbf{D}$ 4x3 3x3 — 4x3

$\mathbf{M}^T\mathbf{H}^T$ 4x2 2x2 — 4x2

AW 3x3 3x4 — 3x4

HM 2x2 2x4 — 2x4

WD 3x4 3x3

MH 2x4 2x2

DW 3x3 3x4 — 3x4

- Compute the following matrix products:

HM and **AD**

$$HM = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 8 & -2 & 5 \\ 2 & 8 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 8 & -5 & 3 \\ 7 & 32 & 1 & 26 \end{pmatrix}$$

$$AD = \begin{pmatrix} 1 & 3 & 8 \\ 3 & 0 & -2 \\ 8 & -2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 15 & 24 \\ 3 & 0 & -6 \\ 8 & -10 & -9 \end{pmatrix}$$

- From the previous computation, **AD**, do you notice anything interesting about multiplying a matrix by a diagonal matrix on the right? Can you generalize what happens in words?

The columns of the matrix A will always be scaled by the corresponding diagonal element of D.

Different Views of Matrix Multiplication

2. Consider the matrix product **AB** where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

Let $\mathbf{C} = \mathbf{AB}$.

- Compute the matrix product **C**.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 10 & 27 \end{pmatrix}$$

- Compute the matrix-vector product $\mathbf{AB}_{\star 1}$ and show that this is the first column of **C**. (Likewise, $\mathbf{AB}_{\star 2}$ is the second column of **C**.) (*Matrix multiplication can be viewed as a collection of matrix-vector products.*)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$$

Likewise $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 27 \end{pmatrix}$

- Compute the two outer products using columns of **A** and rows of **B** and show that

$$\mathbf{A}_{\star 1}\mathbf{B}_{1\star} + \mathbf{A}_{\star 2}\mathbf{B}_{2\star} = \mathbf{C}$$

(*Matrix multiplication can be viewed as the sum of outer products.*)

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}\begin{pmatrix} 2 & 5 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 6 & 15 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ 4 & 12 \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 10 & 27 \end{pmatrix}$$

- Since $\mathbf{AB}_{\star 1}$ is the first column of **C**, show how $\mathbf{C}_{\star 1}$ can be written as a linear combination of columns of **A**. (*Matrix multiplication can be viewed as a collection of linear combinations of columns of the first matrix.*)

$$C_{*1} = B_{11} A_{*1} + B_{22} A_{*2}$$

$$\begin{pmatrix} 4 \\ 10 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

- Finally, note that $\mathbf{A}_{1\star}\mathbf{B}$ will give the first row of **C**. (*This amounts to a linear combination of rows - can you see that?*)

$$\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 11 \end{pmatrix}$$

$$\Rightarrow 1 \cdot \begin{pmatrix} 2 & 5 \end{pmatrix} + 2\begin{pmatrix} 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 11 \end{pmatrix}$$ Linear comb. of rows

$$A_{11} B_{1*} + A_{12} B_{2*} = C_{1*}$$