Worksheet - Lecture 8

Characterizing Solution Sets

1. For each of the following systems, determine whether they have a unique solution, no solution, or infinitely many solutions. If a system has a unique solution, provide that solution. If a system has infinitely many solutions, characterize the solution set as was done in the lecture.

a. $\begin{cases} x + 2y + z = 2 \\ 2x + 4y = 2 \\ 3x + 6y + z = 4 \end{cases}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 4 & 0 & 2 \\ 3 & 6 & 1 & 4 \end{array}\right) \xrightarrow[R_3' = R_3 - 3R_1]{R_2' = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

$$\xrightarrow[R_2' = -\frac{1}{2}R_2]{R_3' = R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1' = R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + 2x_2 =$
$x_3 = 1$

$x_1 = 1 - 2x_2$
$x_2 \equiv$ free
$x_3 = 1$

let $x_2 = s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1-2s \\ s \\ 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}$$

b. $\begin{cases} 2x_1 + 2x_2 + 6x_3 = 4 \\ 2x_1 + x_2 + 7x_3 = 6 \\ -2x_1 - 6x_2 - 7x_3 = -1 \end{cases}$

$$\left(\begin{array}{ccc|c} 2 & 2 & 6 & 4 \\ 2 & 1 & 7 & 6 \\ -2 & -6 & -7 & -1 \end{array}\right)$$

unique solution: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

c. $\begin{cases} -h + l + w = 1 \\ -h - l + w = 2 \\ h - l + w = 3 \\ h + l + w = 4 \end{cases}$

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 2 \\ 1 & -1 & 1 & 3 \\ 1 & 1 & 1 & 4 \end{array}\right) \xrightarrow[\substack{R_3' = R_3 + R_1 \\ R_4' = R_4 + R_1}]{R_2' = R_2 - R_1} \left(\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 2 & 2 & 5 \end{array}\right) \xrightarrow{R_4' = R_4 + R_2} \left(\begin{array}{ccc|c} -1 & 1 & 1 & \\ 0 & -2 & 0 & \\ 0 & 0 & 2 & \\ 0 & 0 & 2 & \end{array}\right)$$

$$\xrightarrow{R_4' = R_4 - R_3} \left(\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{array}\right) \Rightarrow$$ INCONSISTENT

d. $\begin{cases} -h + l + w = 1 \\ -h - l + w = 2 \\ h - l + w = 3 \\ h + l + w = 2 \end{cases}$

same steps as above, only last row changes.
reach $\left(\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 2 \end{pmatrix}$ unique solution

c. $\begin{cases} x_1 + 2x_2 + 2x_3 + 3x_4 = 0 \\ 2x_1 + 4x_2 + x_3 + 3x_4 = 0 \\ 3x_1 + 6x_2 + x_3 + 4x_4 = 0 \end{cases}$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 1 & 4 & 0 \end{array}\right) \xrightarrow[R_3' = R_3 - 3R_1]{R_2' = R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & -3 & -3 & 0 \\ 0 & 0 & -5 & -5 & 0 \end{array}\right)$$

$$\xrightarrow[R_3' = -\frac{1}{5}R_3]{R_2' = -\frac{1}{3}R_2} \left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3' = R_3 - R_2} \left(\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1' = R_1 - 2R_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = -2x_2 - x_4$
$x_2 =$ free
$x_3 = -x_4$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$